2006 Paper 9 Question 11

Types

Given any polymorphic lambda calculus (PLC) type τ and any function ρ mapping type variables α to values $n \in \{-1, 0, 1\}$, a value $[\![\tau]\!]\rho$ in $\{-1, 0, 1\}$ is defined by recursion on the structure of τ as follows.

$[\![\alpha]\!]\rho = \rho(\alpha)$

$$[\![\tau_1 \to \tau_2]\!]\rho = \begin{cases} 1 & \text{if } [\![\tau_1]\!]\rho \leq [\![\tau_2]\!]\rho \\ [\![\tau_2]\!]\rho & \text{otherwise} \end{cases}$$

$[\![\forall\alpha(\tau)]\!]\rho =$ the minimum of the values $[\![\tau]\!](\rho[\alpha \mapsto n])$ for $n = -1, 0, 1$ (where $\rho[\alpha \mapsto n]$ is the function mapping α to n and every other α' to $\rho(\alpha')$).

If Γ is a non-empty PLC typing environment, let $[\![\Gamma]\!]\rho$ denote the minimum value of $[\![\tau]\!]\rho$ as τ ranges over the types in Γ; in the case that Γ is empty, we define $[\![\Gamma]\!]\rho$ to be 1.

(*a*) Prove that if $\Gamma \vdash M : \tau$ is a valid PLC typing judgement, then for any ρ, $[\![\Gamma]\!]\rho \leq [\![\tau]\!]\rho$.

You may assume without proof that if α is not free in τ then

$$[\![\tau]\!](\rho[\alpha \mapsto n]) = [\![\tau]\!]\rho$$

and also that type substitutions $\tau[\tau'/\alpha]$ satisfy

$$[\![\tau[\tau'/\alpha]]\!]\rho = [\![\tau]\!](\rho[\alpha \mapsto [\![\tau']\!]\rho])$$

[Hint: show that the property $\Phi(\Gamma, M, \tau) =$ "for all ρ, $[\![\Gamma]\!]\rho \leq [\![\tau]\!]\rho$" is closed under the rules of the typing system.]

[16 marks]

(*b*) Deduce that there is no closed PLC expression of type

$$\forall\alpha, \beta(((\alpha \to \beta) \to \alpha) \to \alpha)$$

[4 marks]