2007 Paper 2 Question 5

Discrete Mathematics II

The purpose of this question is to look at a method for counting certain finite sets that arise as quotients under an equivalence relation; and to apply the method to count the number of injections between two finite sets.

For a set A, let $\mathsf{Bij}(A)$ be the set of bijections from A to A. An A-*action* on a set X is defined to be a function $\star : X \times \mathsf{Bij}(A) \to X$, typically written in infix notation so that $x \star \sigma = \star(x, \sigma)$, such that $x \star \mathsf{id}_A = x$ and $(x \star \sigma) \star \tau = x \star (\sigma \circ \tau)$ for all $x \in X$ and $\sigma, \tau \in \mathsf{Bij}(A)$.

(a) Let $\star : X \times \mathsf{Bij}(A) \to X$ be an A-action on X. Show that the relation $\sim$ on X defined by $x \sim y \stackrel{\text{def}}{\iff} \exists \sigma \in \mathsf{Bij}(A).\ x = y \star \sigma$, for all $x, y \in X$, is an equivalence relation. [6 marks]

(b) As usual, let $[x]_\sim \stackrel{\text{def}}{=} \{\, y \in X \mid x \sim y \,\}$ be the equivalence class of $x \in X$ under the equivalence relation $\sim$. Furthermore, for $x \in X$, let $e_x : \mathsf{Bij}(A) \to [x]_\sim$ be the function defined by $e_x(\sigma) \stackrel{\text{def}}{=} x \star \sigma$, for all $\sigma \in \mathsf{Bij}(A)$.

For $x \in X$, prove that e_x is surjective. For A a finite set of size n, what does this tell us about the size of $[x]_\sim$, for $x \in X$? [2 marks]

(c) An A-action on X is said to be *faithful* if the function e_x is injective for all $x \in X$. In this case:

(i) If A is a finite set of size n, what is the size of each $[x]_\sim$, for $x \in X$?

(ii) If, in addition, X is a finite set of size m, what is the size of the set of equivalence classes $X/_\sim \stackrel{\text{def}}{=} \{\, [x]_\sim \mid x \in X \,\}$?

Justify your answers. [6 marks]

(d) For sets A and B, let $\mathsf{Inj}(A, B)$ be the set of injections from A to B. Show that the function $\bullet : \mathsf{Inj}(A, B) \times \mathsf{Bij}(A) \to \mathsf{Inj}(A, B)$ defined by $\iota \bullet \sigma \stackrel{\text{def}}{=} \iota \circ \sigma$, for all $\iota \in \mathsf{Inj}(A, B)$ and $\sigma \in \mathsf{Bij}(A)$, is an A-action on $\mathsf{Inj}(A, B)$. Prove also that it is faithful. [6 marks]

Note that since $\mathsf{Inj}(A, B)/_\sim \cong \{\, S \subseteq B \mid S \cong A \,\}$, it follows from the results in (c)(ii) and (d) that, for A and B finite, $\#\big(\mathsf{Inj}(A, B)\big) = \binom{\#B}{\#A}(\#A)!$.