2007 Paper 4 Question 2

Probability

Suppose you have k light bulbs, where $k > 1$, and that the probability of any individual bulb not working is p. Two strategies for testing the k bulbs are:

(A) Test each bulb separately. This takes k tests.

(B) Wire up all k bulbs as a series circuit. If all the bulbs come on, the testing is complete in just one test, otherwise revert to strategy A taking a total of $k+1$ tests.

Let X be a random variable whose value r is the number of tests required using strategy B. The probability $\mathrm{P}(X = r)$ may be expressed as:

$$\mathrm{P}(X = r) = \begin{cases} (1-p)^k, & \text{if } r = 1 \\ 1-(1-p)^k, & \text{if } r = k+1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Explain this function and justify the constraint $k > 1$. [4 marks]

(b) Determine the Expectation $\mathrm{E}(X)$. [4 marks]

(c) Strategy B beats strategy A (by requiring fewer tests) if $\mathrm{E}(X) < k$ and this condition is satisfied if $p < f(k)$ where $f(k)$ is some function of k. Derive the function $f(k)$. [8 marks]

[Note that $f(k) \to 0$ as $k \to \infty$ and that the maximum value of $f(k) \approx 0.307$ (when $k = 3$). Strategy B is therefore never useful if $p > 0.307$.]

(d) Suppose you have n light bulbs, where $n \gg k$ and k divides n so that $n = m.k$, and you partition the n bulbs into m groups of k. Assuming that the groups are independent and again assuming that $k > 1$, show that the expected number of tests is:

$$n\big[1 + \frac{1}{k} - (1-p)^k\big].$$

Give a rough description of how, for a given value of p, the expression in square brackets varies with k and suggest how someone responsible for testing light bulbs might exploit this expression. [4 marks]