4 Computation Theory (AMP)

(*a*) Give inductive definitions of the relations $M \to N$ and $M \twoheadrightarrow N$ of *single-step* and *many-step β-reduction* between λ-terms M and N. (You may assume the definition of α-conversion, $M =_\alpha N$.) [6 marks]

(*b*) *Turing's fixed point combinator* is the λ-term $\mathtt{A}\,\mathtt{A}$ where $\mathtt{A} = \lambda x.\lambda y.\, y(x\,x\,y)$. Use it to show that given any λ-term M, there is a λ-term X satisfying $X \twoheadrightarrow M\,X$. [2 marks]

(*c*) The sequence of λ-terms $\mathtt{N}_0, \mathtt{N}_1, \mathtt{N}_2, \ldots$ is defined by $\mathtt{N}_0 = \lambda x.\lambda f.\, x$ and $\mathtt{N}_{n+1} = \lambda x.\lambda f.\, f\,\mathtt{N}_n$. Say that a function $\mathtt{f} \in \mathbb{N}^k \to \mathbb{N}$ is *Scott definable* if there is a λ-term F satisfying that $F\,\mathtt{N}_{n_1} \cdots \mathtt{N}_{n_k} \twoheadrightarrow \mathtt{N}_{\mathtt{f}(n_1,\ldots,n_k)}$ for all $(n_1, \ldots, n_k) \in \mathbb{N}^k$.

(*i*) Show that the successor function , $\mathtt{succ}(n) = n + 1$, is Scott definable. [2 marks]

(*ii*) Show that for any λ-terms M and N, $\mathtt{N}_0\,M\,N \twoheadrightarrow M$ and $\mathtt{N}_{n+1}\,M\,N \twoheadrightarrow N\,\mathtt{N}_n$. Deduce that the predecessor function

$$\mathtt{pred}(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{if } n > 0 \end{cases}$$

is Scott definable. [2 marks]

(*iii*) By considering the λ-terms $P_m = \mathtt{A}\,\mathtt{A}\,(\lambda f.\lambda y.\, y\,\mathtt{N}_m\,(\lambda z.\, S(f\,z)))$ for a suitable choice of S, or otherwise, prove that the addition function $\mathtt{plus}(m, n) = m + n$ is Scott definable. [8 marks]