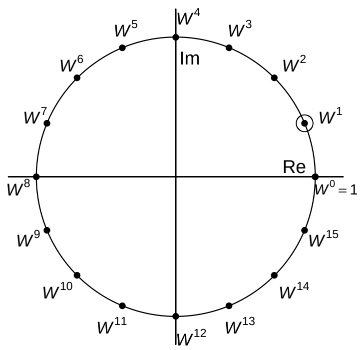
7 Mathematical Methods for Computer Science (JGD)

(*a*) An inner product space E containing piecewise continuous complex functions $f(x)$ and $g(x)$ on some interval is spanned by the orthonormal basis functions $\{e_i\}$ used in the Fourier series. Thus complex coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ exist such that $f(x) = \sum_i \alpha_i e_i(x)$ and $g(x) = \sum_i \beta_i e_i(x)$.

(*i*) Show that $\langle f, g \rangle = \sum_i \alpha_i \overline{\beta_i}$. [5 marks]

(*ii*) Would the same result hold if the orthonormal basis functions $\{e_i\}$ that span E were *not* the Fourier basis? Justify your answer, and provide the name for coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ in such a case. [2 marks]

(*b*) Consider a sequence $f[n]$ ($n = 0, 1, \ldots, 15$) with Fourier coefficients $F[k]$ ($k = 0, 1, \ldots, 15$). Using the 16^{th} roots of unity as labelled around the unit circle in powers of w^1, the primitive 16^{th} root of unity, construct a sequence of these w^i that could be used to compute $F[3]$.



[4 marks]

(*c*) From the well-known fact that a periodic square wave ($f(x) = 1$ for $0 < x < \pi$, $f(x) = -1$ for $\pi < x < 2\pi$, $\cdots$) has the following Fourier series

$$f(x) = \frac{4}{\pi}\left[\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \cdots\right]$$

produce the first four terms of the Fourier series for the triangle wave whose derivative is this square wave. [4 marks]

(*d*) What sets of frequencies are required to perform the following analyses?

- Fourier transform of a non-periodic continuous function
- Fourier analysis of a piecewise continuous periodic function with period 2π
- Wavelet transform of a non-periodic function, either continuous or discrete

Comment on the relationship between the density of frequencies required and the role of "locality" in the analysis. [5 marks]