8 Mathematical Methods for Computer Science (RJG)

(*a*) Let X be a random variable with finite mean $\mu = \mathbb{E}(X)$ and finite variance $\sigma^2 = \text{Var}(X)$. State and prove Chebyshev's inequality for the random variable X. You may assume Markov's inequality without proof. [5 marks]

(*b*) Now suppose that X is a continuous random variable with probability density function $f_X(x)$ and finite mean $\mu = \mathbb{E}(X)$ such that

- $f_X(x) = 0 \quad \forall x \notin [\alpha, \beta]$
- $x f_X(x) \le \gamma \quad \forall x \in [\alpha, \beta]$

where α, β and γ are non-negative real constants with $\alpha < \beta$. Suppose that (A_i, B_i) for $i = 1, 2, \ldots, n$ is a sequence of independent and identically distributed 2-dimensional random variables where A_i and B_i are independent with marginal distributions $A_i \sim U[\alpha, \beta]$ and $B_i \sim U[0, \gamma]$ for each $i = 1, 2, \ldots, n$.

(*i*) Define random variables I_i for $i = 1, 2, \ldots, n$ such that

$$I_i = \begin{cases} 1 & \text{if } B_i \le A_i f_X(A_i) \\ 0 & \text{otherwise} \end{cases}$$

and set $Z_n = \frac{1}{n}\sum_{i=1}^{n} I_i$. Show that $\mathbb{E}(Z_n) = \mu/(\gamma(\beta - \alpha))$ and that $\text{Var}(Z_n) \le \frac{1}{4n}$. [5 marks]

(*ii*) Using Chebyshev's inequality show that Z_n converges in probability to the degenerate random variable with value $\mu/(\gamma(\beta - \alpha))$. [5 marks]

(*iii*) Describe an algorithm to estimate the mean μ of the random variable X. You may assume for the purpose of your algorithm that you have a function that returns random points of the given form (A_i, B_i). [5 marks]