7 Mathematical Methods for Computer Science (JGD)

(*a*) For an inner product space $V = \mathbb{R}^4$ with Euclidean norm and a set of vectors $\{w, x, y, z\} \in V$

$$w = \left(-\tfrac{1}{2},\ \tfrac{1}{2},\ -\tfrac{1}{2},\ \tfrac{1}{2}\right) \qquad x = \left(-\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ -\tfrac{1}{2}\right)$$

$$y = \left(\tfrac{1}{2},\ \tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right) \qquad z = \left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right)$$

demonstrate both whether the vectors $\{x, y, z, w\}$ form an *orthogonal system*, and whether they form an *orthonormal system*. [4 marks]

(*b*) Let $f[n]$ for $n = 0, 1, 2, \cdots, N-1$ be a discrete data sequence of N complex values, where $N = 2^Z$ is some integer power of 2.

(*i*) Define the Discrete Fourier Transform (DFT) of $f[n]$, a discrete sequence $F[k]$, as a series using complex exponentials. In terms of N, how many complex multiplications would be required to compute $F[k]$ explicitly? Then define W, the primitive N^{th} root of unity, and now re-express your series for $F[k]$ in terms of W. [4 marks]

(*ii*) Now express your DFT sequence $F[k]$ using a series with only $N/2$ terms, by capturing the second half of the series within the first half, and show that fewer complex multiplications are required. [4 marks]

(*iii*) Now show that separating $F[k]$ into two new sequences of half-length, each of which computes only $N/2$ coefficients $F[k]$, leads to a very efficient recursive form. How many times can this halving process be repeated? Ultimately how many complex multiplications are therefore required for this Fast Fourier Transform (FFT)? For 'big data' applications requiring a DFT on a billion data values, what speed-up factor can be expected by using an FFT instead of explicitly computing a DFT? [4 marks]

(*c*) Piecewise continuous and absolutely integrable functions $f(x), g(x) : \mathbb{R} \to \mathbb{C}$ have Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. Let $h(x) = (f * g)(x)$ be the convolution of $f(x)$ and $g(x)$:

$$h(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy$$

Prove that $H(\omega)$, the Fourier transform of $h(x)$, is simply the product:

$$H(\omega) = 2\pi F(\omega) \cdot G(\omega)$$

[4 marks]