4 Complexity Theory (ad260)

For the purpose of this question, a graph $G = (V, E)$ is a set V of vertices along with a set E of edges where each edge is a set of two *distinct* vertices. That is, we consider undirected graphs without self-loops or multiple edges.

Given two graphs $G = (V, E)$ and $H = (U, F)$, a *homomorphism* from G to H is a function $h : V \to U$ such that whenever $\{v_1, v_2\}$ is in E, $\{h(v_1), h(v_2)\}$ is in F. We write HOM for the decision problem consisting of all pairs of graphs (G, H) such that there is a homomorphism from G to H.

Recall that a graph $G = (V, E)$ is k-colourable (for a positive integer k) if there is a function $\chi : V \to \{1, \ldots, k\}$ such that whenever $\{u, v\}$ is in E, $\chi(u) \neq \chi(v)$.

(*a*) Explain why the decision problem HOM is in NP. [4 marks]

(*b*) Let K_3 denote the graph with three vertices a, b, c and the three edges $\{a, b\}, \{b, c\}$ and $\{a, c\}$. Show that for any graph G, there is a homomorphism from G to K_3 if, and only if, G is 3-colourable. [6 marks]

(*c*) What can you conclude from the above about the complexity of the problem HOM? [5 marks]

(*d*) Let K_2 denote the graph consisting of two vertices a and b and the single edge $\{a, b\}$. What is the complexity of the decision problem consisting of all graphs G for which there is a homomorphism from G to K_2? [5 marks]