12 Randomised Algorithms (tms41)

(*a*) State the main advantages and disadvantages of Chernoff bounds in comparison to Markov's inequality and Chebyshev's inequality. [3 marks]

(*b*) Let $X_1, X_2, \ldots, X_n$ be independent random variables with $\mathbf{P}[\, X_i = -1\,] = \mathbf{P}[\, X_i = 1\,] = 1/2$ for $i = 1, 2, \ldots, n$. Let $X := \sum_{i=1}^{n} X_i$.

(*i*) What is $\mathbf{E}[\, X\,]$? [1 mark]

(*ii*) Derive the following concentration inequality: For any $t > 0$,

$$\mathbf{P}[\, X > t\,] \leq e^{-t^2/(2n)}.$$

Hint: You may use the inequality $\mathbf{E}\left[\, e^{\lambda X_i}\,\right] \leq e^{\lambda^2/2}$, which holds for any $1 \leq i \leq n$ and $\lambda > 0$. [5 marks]

(*c*) Consider now a random assignment of m jobs to n processors, such that each job is assigned to a processor chosen independently and uniformly at random from $\{1, 2, \ldots, n\}$. Independently of this assignment, each job takes 1 time unit to complete with probability 1/2, and 3 time units otherwise.

(*i*) What is the expected number of time units assigned to one processor? [2 marks]

(*ii*) Derive an upper bound on the total sum over the time units of the m jobs that holds with probability $1 - 1/m$.
Hint: You may use the result from (*b*)(*ii*). [4 marks]

(*iii*) Derive a concentration inequality for the number of processors which are assigned a number of time units that is at least as large as the expected number of time units per processor. For full marks, you should define the involved random variables and quantities precisely, and consider both the lower and upper tail. [5 marks]