

4 Computation Theory (amp12)

(a) For the λ -calculus, define the notions of

(i) β -conversion ($=_\beta$) [2 marks]

(ii) Church numeral (\underline{n}) [2 marks]

(b) What does it mean for a total function $f : \mathbb{N}^n \rightarrow \mathbb{N}$ to be λ -definable? Explain why it is the case that not every $f : \mathbb{N}^n \rightarrow \mathbb{N}$ is λ -definable, carefully stating any standard results that you rely upon. [3 marks]

(c) Explain why the predecessor function $pred : \mathbb{N} \rightarrow \mathbb{N}$

$$pred(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{if } x > 0 \end{cases}$$

is λ -definable and give a λ -term that represents it. [4 marks]

(d) Show that the following functions are λ -definable. For each part you may assume solutions to the previous parts of the question.

(i) $if_0 : \mathbb{N}^3 \rightarrow \mathbb{N}$, where $if_0(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{if } x \neq 0 \end{cases}$ [3 marks]

(ii) $and : \mathbb{N}^2 \rightarrow \mathbb{N}$, where $and(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 0 \\ 1 & \text{if } x \neq 0 \text{ or } y \neq 0 \end{cases}$ [1 mark]

(iii) $monus : \mathbb{N}^2 \rightarrow \mathbb{N}$, where $monus(x, y) = \begin{cases} x - y & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$ [3 marks]

(iv) $eq : \mathbb{N}^2 \rightarrow \mathbb{N}$, where $eq(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ [2 marks]