COMPUTER SCIENCE TRIPOS Part II – 2023 – Paper 8

12 Randomised Algorithms (tms41)

In this question, all considered graphs are undirected and d-regular.

(a) State the definition of conductance. [2 marks]

(b) If G is disconnected, what does this imply in terms of the conductance? [1 mark]

(c) If G is disconnected, what does this imply in terms of the spectrum of $\mathbf{L}$? Briefly justify your claim. [3 marks]

(d) The d-dimensional hypercube with $n = 2^d$ vertices is defined by creating a vertex for each d-digit binary number $(x_1, x_2, \ldots, x_d) \in \{0,1\}^d$. Further, any two vertices are adjacent if and only if their binary representations differ in exactly one digit.

(i) Identifying each binary representation $(x_1, x_2, \ldots, x_d) \in \{0,1\}^d$ with a unique vertex label in $\{1, 2, \ldots, n\}$, verify that $f_{(x_1,x_2,\ldots,x_d)} = (-1)^{x_1}$ is an eigenvector of the Laplacian Matrix $\mathbf{L}$ of the hypercube. State the associated eigenvalue of f for both $\mathbf{L}$ and the adjacency matrix $\mathbf{A}$. [7 marks]

(ii) Apply the Spectral Clustering Algorithm to estimate the conductance of the hypercube, assuming that f in (d)(i) is the eigenvector of λ_2. [*Hint:* It suffices to apply the $(n/2)$-th sweep cut only.] [5 marks]

(iii) Combining (d)(ii) and (d)(i), what can you conclude about the found cut? [2 marks]