## 4 Computation Theory (tgg22)

These questions deal with the theory of partial recursive functions.

Given a subset $S \subseteq \mathbb{N}$, its characteristic function $\chi_S \in \mathbb{N} \to \mathbb{N}$ is given by

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & z \notin S. \end{cases}$$

We say that $S$ is *decidable* if $\chi_S$ is a total recursive function.

A set $S$ will be called *recursively enumerable* if it is empty or there is a total recursive function $f$ such that

$$S = \{f(n) \mid n \in \mathbb{N}\}.$$

Note that it may be that $f(n) = f(m)$ for some $m \neq n$. That is, the enumeration may involve repeats.

(*a*) Are all total partial recursive functions primitive recursive? [2 marks]

(*b*) Suppose that $S \subseteq \mathbb{N}$ is decidable. Prove that $S$ is recursively enumerable. [6 marks]

(*c*) Prove that a set $S \subseteq \mathbb{N}$ is decidable if and only if both $S$ and its complement $\overline{S}$ are recursively enumerable. [6 marks]

(*d*) Provide a complete proof that the set $S_0 = \{e \mid \phi_e(0)\downarrow\}$ is recursively enumerable but its compliment is not. [6 marks]