5 Denotational Semantics (mgapb2)

We consider the language called System T, whose types and terms are respectively given by the following grammars:

$$\tau := \mathtt{nat} \mid \tau \to \tau$$
$$t := \mathtt{0} \mid \mathtt{succ}(t) \mid \mathtt{iter}(t, t, t) \mid x \mid \mathtt{fun}\, x \colon \tau.\ t \mid t\ t$$

Contexts, as in PCF, are partial maps from variables to types. Typing and operational semantics is the same as in PCF, except for iteration, which is as follows:

$$\frac{\Gamma \vdash n : \mathtt{nat} \quad \Gamma \vdash t_0 : T \quad \Gamma \vdash t_{\mathtt{succ}} : T \to T}{\Gamma \vdash \mathtt{iter}(n, t_0, t_{\mathtt{succ}}) : T}$$

$$\frac{n \Downarrow_{\mathtt{nat}} \mathtt{0} \quad t_0 \Downarrow_\tau v}{\mathtt{iter}(n, t_0, t_{\mathtt{succ}}) \Downarrow_\tau v} \qquad \frac{n \Downarrow_{\mathtt{nat}} \mathtt{succ}(n') \qquad t_{\mathtt{succ}}\ \ \mathtt{iter}(n', t_0, t_{\mathtt{succ}}) \Downarrow_\tau v}{\mathtt{iter}(n, t_0, t_{\mathtt{succ}}) \Downarrow_\tau v}$$

We keep the PCF rules for the operational semantics of values, successor and function application.

In System T, we only include bounded iteration, so all functions are total. The goal of this question is to give a denotational semantics to System T using **sets** and **total functions** rather than domains to reflect this.

(*a*) Give the PCF (and System T) typing rules for 0, `succ`, variables, `fun` and application, and the operational semantic rules for values, successor and application. [3 marks]

(*b*) Give a denotation $[\![\cdot]\!]$ to System T types and contexts, such that if τ is a type then $[\![\tau]\!]$ is a set. [2 marks]

(*c*) Given the answer to (*b*), state what should be the interpretation of a System T term t such that $\Gamma \vdash t : \tau$. [2 marks]

(*d*) Give a denotational semantics $[\![\cdot]\!]$ for (well-typed) System T terms. Justify that this denotation is well-defined. [5 marks]

(*e*) State what it means for this denotation to be sound. Show that this is indeed the case. You can freely assume that the denotation is substitutive, provided you clearly state that property. [5 marks]

(*f*) Let $\overline{0} \in \mathbb{N} \to \mathbb{N}$ be the constant 0 function, and

$$\begin{array}{rcll} \text{is\_zeroes} \in & (\mathbb{N} \to \mathbb{N}) & \to & \mathbb{N} \\ & \overline{0} & \mapsto & 0 \\ & f & \mapsto & 1 \qquad \text{otherwise} \end{array}$$

Do you think is_zeroes is definable with respect to the semantics from part (*d*)? Explain informally why, or why not, and what could be a strategy to prove this fact. You do *not* need to give a full proof. [3 marks]