Technical Report

UCAM-CL-TR-346
ISSN 1476-2986

Number 346

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

On intuitionistic linear logic

G.M. Bierman

August 1994

15 JJ Thomson Avenue
Cambridge CB3 0FD
United Kingdom
phone +44 1223 763500

https://www.cl.cam.ac.uk/

This technical report is based on a dissertation submitted December 1993 by the author for the degree of Doctor of Philosophy to the University of Cambridge, Wolfson College.

Technical reports published by the University of Cambridge Computer Laboratory are freely available via the Internet:

https://www.cl.cam.ac.uk/techreports/

ISSN 1476-2986

Summary

In this thesis we carry out a detailed study of the (propositional) intuitionistic fragment of Girard's linear logic (**ILL**). Firstly we give sequent calculus, natural deduction and axiomatic formulations of **ILL**. In particular our natural deduction is different from others and has important properties, such as closure under substitution, which others lack. We also study the process of reduction in all three logical formulations, including a detailed proof of cut elimination. Finally, we consider translations between Intuitionistic Logic (**IL**) and **ILL**.

We then consider the linear term calculus, which arises from applying the Curry-Howard correspondence to the natural deduction formulation. We show how the various proof theoretic formulations suggest reductions at the level of terms. The properties of strong normalization and confluence are proved for these reduction rules. We also consider mappings between the extended λ-calculus and the linear term calculus.

Next we consider a categorical model for **ILL**. We show how by considering the linear term calculus as an equational logic, we can derive a model: a *Linear category*. We consider two alternative models: firstly, one due to Seely and then one due to Lafont. Surprisingly, we find that Seely's model is *not* sound, in that equal terms are not modelled with equal morphisms. We show how after adapting Seely's model (by viewing it in a more abstract setting) it becomes a particular instance of a linear category. We show how Lafont's model can also be seen as another particular instance of a linear category. Finally we consider various categories of coalgebras, whose construction can be seen as a categorical equivalent of the translation of **IL** into **ILL**.

Acknowledgements

Firstly, and above all, I should like to thank Valeria de Paiva. Without her guidance and help this thesis would never have been written. I cannot think of a time when she was not willing to discuss my work. In particular, her mathematical expertise has been a source of inspiration. It has also been my good fortune to work with Martin Hyland on some of the matters in this thesis and I have benefited from his unique insight and generous assistance. I have discussed with Nick Benton many matters of theoretical computer science and most of the contents of this thesis. I have learnt a great deal from him. Alan Mycroft has offered much sound advice and encouragement.

I have also benefited from discussions with (amongst others) Torben Brauner, Paul Curzon, Francis Davey, Barney Hilken, Ian Mackie, Luke Ong, Andy Pitts, Eike Ritter, Harold Schellinx, Ian Stark, Anne Troelstra and Phil Wadler.

I should also like to thank Nick, Paul, Martin, Valeria and Eike for commenting on various drafts of this work and making many helpful suggestions. In particular, Nick made some insightful comments on some of the categorical work. I have used Paul Taylor's TEX macros and Vaughan Pratt's 'Par' macro in preparing this thesis.

In November 1992 I was invited to spend a week at Århus University. I learnt a great deal from this visit and it shaped this thesis. I should like to thank Glynn Winskel for inviting me and Torben Brauner and Bettina Blaaberg for their hospitality.

I have received financial support from SERC, Wolfson College, the Computer Laboratory, the Cambridge Philosophical Society and from the CLiCS BRA. I am grateful to these bodies for their support.

Finally I should like to thank my parents and Cathy Head for their love and support.

Preface to Technical Report Version.

This technical report is a revised version of my thesis, submitted in December 1993 and examined in April 1994. I am grateful to my examiners, Dr A.M. Pitts and Prof. G.D. Plotkin, for their helpful comments and useful suggestions on this work. Any mistakes that remain are my sole responsibility.

Contents

List of Figures

Chapter 1
Introduction

1 Background

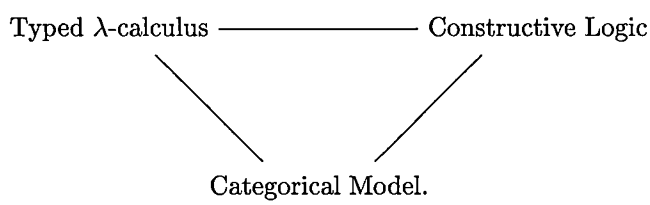
An important problem in theoretical computer science is discovering logical foundations of programming languages. Such foundations provide programmers with techniques for reasoning logically rather than informally about their programs and not only tells implementors precisely what they are trying to implement but also enables them to reason formally about possible optimizations. One of the most fruitful methods used to explore such logical foundations has been to utilize a fascinating relationship between various typed λ-calculi; constructive logics and structures, or models, from category theory. Despite their apparent independence, various work has shown how these areas are related.

Logic with typed λ-calculi. Curry [23, Section 9E] and Howard [41] noticed that an axiomatic formulation of Intuitionistic Logic (**IL**) corresponds to the type scheme for (S,K) combinatory logic. It was also noted that the natural deduction formulation for the $(\supset, \wedge)$-fragment of minimal logic corresponds to the typing rules for the simply typed λ-calculus with pairs. More importantly, the notion of normalization for minimal logic corresponds to the notion of reduction of the λ-terms. This relationship is known as the 'propositions-as-types analogy' or the *Curry-Howard correspondence.*

Category theory with logic. Lambek [48, 49, 50] first showed how formal deductions for propositional logics could be given in a categorical framework. In particular he considered the relationship between intuitionistic (propositional) logic and cartesian closed categories. Lawvere [54] showed how these techniques could be extended to handle predicate logics by considering more powerful categorical structures.

Category theory with typed λ-calculi. Given the relationship between logic and typed λ-calculi, Lambek was able to show how category theory could give a semantics for typed λ-calculi; this is demonstrated for various calculi in his book with Scott [52]. Curien, along with co-workers, has shown how this semantics can be seen to suggest an abstract machine: the *categorical abstract machine* [21]. This idea has been used to provide a complete compiler for a dialect of ML [72].

These relationships can be pictured as



The exciting aspect of these relationships is not just that the three areas are related but that certain concepts within them are related also, as shown below.

Logic	Typed λ-calculus	Categorical Model
Proposition	Type	Object
Proof	Term	Morphism
Normalization	Reduction	Equality of morphisms

These relationships have been used to study various systems. Some examples are listed below.

Logic	Typed Calculus	Categorical Model
Intuitionistic Logic	Simply Typed λ-calculus	Cartesian Closed Categories
Second Order Intuitionistic Logic	System F [32]	Hyperdoctrines [68]
Higher Order Intuitionistic Logic	Higher Order Typed λ-calculus	Toposes [52]

This thesis explores these three relationships and related concepts with respect to *Intuitionistic Linear Logic* (**ILL**) (in fact, just the propositional part). We take **ILL** as described by Girard and after producing natural deduction, sequent calculus and axiomatic presentations we consider the corresponding term calculus (the *linear term calculus*) and categorical model (a *linear category*).

2 Overview of Logical Systems and The Curry-Howard Correspondence

In this thesis we shall consider three logical systems in which to formulate **ILL**: natural deduction, sequent calculus and axiomatic system. Let us briefly consider these systems in turn.

Natural deduction was originally proposed by Gentzen [73]. Deductions proceed in a tree-like manner where a conclusion is derived from a finite number of assumption packets, using a predefined set of inference rules. More specifically these packets contain a multiset of propositions and may be empty. Within a deduction we may 'discharge' any number of assumption packets. This discharging of packets can be recorded in one of two ways. Gentzen proposed annotating assumption packets with labels (natural numbers). Occurrences of inference rules which discharge packets are then annotated with the labels of the packets they discharge. For example the inference rule for the introduction of an implication is as follows:

$$\frac{\begin{array}{c}[A^x]\\ \vdots\\ B\end{array}}{A \supset B}\,(\supset_{\mathcal{I}})_x$$

The square brackets identify that the packet (with label x) has been discharged. Typically we provide rules for the *introduction* and *elimination* of the logical connectives. For example, we provide the following elimination rule for the implication connective.

$$\frac{\begin{array}{c}\vdots\\ A \supset B\end{array} \qquad \begin{array}{c}\vdots\\ A\end{array}}{B}\,(\supset_{\mathcal{E}})$$

The second alternative for annotations is to place at every stage of the deduction tree a complete list of the undischarged assumption packets. This we shall refer to as natural deduction in a 'sequent-style'. Deductions are trees whose nodes are decorated with sequents of the form $\Gamma \vdash A$, where Γ represents the undischarged assumption packets and A the deduction so far. The first method is probably the more intuitive whereas the second has a more mathematical feel. Of course, both approaches are equivalent and we shall present both when considering natural deduction formulations of **ILL**.

The Curry-Howard correspondence simply annotates the deduction with a 'term', which represents an encoding of the deduction so far. Thus for each inference rule, we introduce a unique piece of syntax to represent an application of it. For example, the rule for implication introduction becomes encoded as

$$\dfrac{\begin{matrix}[A^x] \\ \vdots \\ M\colon B\end{matrix}}{\lambda x\colon A.M\colon A \supset B}\,(\supset_{\mathcal{I}})_x.$$

It is common to annotate assumption packets with alphabetic identifiers rather than natural numbers. Thus we have a system for term formation, where the terms have the property that they *uniquely* encode the deduction.

Although natural deduction has many compelling qualities, it has several disadvantages of which we shall mention just two (Girard [34, Page 74] gives a fuller criticism). Firstly it is distinctly asymmetric; there is always a single deduction from a number of assumptions.[1] Secondly, some connectives can only be formulated in an unsatisfactory way. Consider the rule for eliminating a disjunction

$$\dfrac{\begin{matrix} \\ \vdots \\ A \vee B\end{matrix} \qquad \begin{matrix}[A^x] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B^y] \\ \vdots \\ C\end{matrix}}{C}\,(\vee_{\mathcal{E}})_{x,y}.$$

The deduction C really has nothing to do with the connective being eliminated at all: it is often dubbed *parasitic.*

The second system we consider is the sequent calculus, again introduced by Gentzen [73]. Deductions consist of trees of sequents of the form $\Gamma \vdash \Delta$, where both Γ and Δ represent collections of propositions. Inference rules introduce connectives on the right and on the left of the 'turnstile' and rules have a more symmetric feel to them. For example the rules for introducing an implication are

$$\dfrac{\Gamma \vdash A, \Delta \qquad B, \Gamma' \vdash \Delta'}{\Gamma, A \supset B, \Gamma' \vdash \Delta, \Delta'}\,(\supset_{\mathcal{L}}), \quad \text{and} \quad \dfrac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash A \supset B, \Delta}\,(\supset_{\mathcal{R}}).$$

In this thesis we will be concerned only with *intuitionistic* logics. These can be obtained by restricting the sequents to at most a single proposition on the right of the turnstile, $\Gamma \vdash A$. There are other, less restrictive ways of formulating intuitionistic logics but we shall not consider them here. (For example, Hyland and de Paiva [43] have proposed a less restrictive (but more powerful) formulation of **ILL**.)

Let us consider the form of Γ in a sequent $\Gamma \vdash A$. We have a choice as to whether it represents a set, multiset or sequence of propositions. Recall that in the natural deduction system, we had multisets of assumptions, which could be empty. As the sequent calculus is an equivalent logical system it must offer similar manipulations of its assumptions. These manipulations are provided by so-called *structural rules*.[2] The structural rules needed depends on the chosen form of contexts and on the way the inference rules are devised (as they can have the effect of structural rules 'built-in'). Generally we take the contexts to be multisets and then we have two structural rules[3]

$$\dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\textit{Weakening}, \quad \text{and} \quad \dfrac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\textit{Contraction}.$$

[1] A number of multiple conclusion formulations of natural deduction have been proposed, but they invariably introduce more problems than they solve.

[2] These rules really exist in the natural deduction formulation as well but they are often either embedded in informal conditions concerning the assumption packets or built into the inference rules.

[3] Were we interested in the order in which assumptions were used, then we would take contexts to be *sequences* of assumptions and have an explicit *Exchange* rule:

$$\dfrac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C}\,\textit{Exchange}$$

The *Weakening* rule allows for the fact that a multiset can be empty and the *Contraction* rule allows for the fact that many packets can have the same label. We shall see later that these rules play a crucial rôle in formulating linear logic.

The sequent calculus has one main defect which is that it distinguishes between many equivalent proofs. For example consider the following two proofs which only differ in the order in which the inference rules are applied.

$$\dfrac{\dfrac{A, C \vdash D}{A \vdash C \supset D}(\supset_{\mathcal{R}})}{A \wedge B \vdash C \supset D}(\wedge_{\mathcal{L}}) \qquad \dfrac{\dfrac{A, C \vdash D}{A \wedge B, C \vdash D}(\wedge_{\mathcal{L}})}{A \wedge B \vdash C \supset D}(\supset_{\mathcal{R}})$$

When we come to relate the sequent calculus and natural deduction formulations it is clear that these two sequent derivations actually represent the same natural deduction derivation. Of course, this means that a Curry-Howard correspondence is not as simple to devise for a sequent calculus formulation. We shall return to this point in Chapter 2.

A third system is an axiomatic formulation (or Hilbert system). With this formulation there are a collection of 'axioms' and rules for combining these axioms. Deductions proceed in a tree-like fashion from a collection of assumptions or axioms to a conclusion. This style of formulation was used to great effect by Russell and Whitehead in their *Principia Mathematica* [65]. The main advantage of an axiomatic formulation is its simplicity. We shall discuss this sort of formulation in more detail in Chapter 2.

3 Overview of Linear Logic

Linear Logic is often described as a 'resource-conscious' logic. In the context of mathematical logic, we can consider a proposition to represent a resource of some kind. As we noted earlier, logical formulations provide structural rules to manipulate assumptions: the *Weakening* and *Contraction* rules. The *Weakening* rule says that if from a collection of assumptions Γ we can conclude B, then certainly from the assumptions Γ *and* A we can conclude B. The *Contraction* rule says that if we need an assumption A twice to conclude B, then we can simplify this to A, as A and A is morally the same as A.

However, if we take a resource view of these rules they seem slightly strange. The *Weakening* rule amounts to saying that we might not need a resource after all; and the *Contraction* rule tells us that we might need a resource any number of times. Linear Logic is the logic obtained by removing these two rules from the formulation of the logic. (There are both intuitionistic and classical formulations of linear logic; this thesis is concerned with the intuitionistic fragment.) As we shall see later, an immediate consequence is to divide the inference rules into two distinct kinds.

Of course, the logic which remains is terribly weak; Girard's innovation was to reintroduce the two rules in a controlled way by introducing a new unary connective (the so-called *exponential*, '!'). Thus we can only weaken or contract a proposition, A, if it is of the form $!A$. It is this which distinguishes linear logic from other 'sub-structural' logics. Although logicians had conceived many years ago relevance logic (no *Weakening*) and affine logic (no *Contraction*), they appeared to have little interest other than as weak forms of logic. The recapturing of logical power by use of the exponential enables us to consider linear logic as a *refinement* of traditional logics.

4 Outline of Thesis

The structure of this thesis follows the triangle of relationships given in §1. Hence there are three chapters corresponding to each vertex of the triangle.

- Chapter 2 considers the proof theory of **ILL**. After giving a sequent calculus formulation, a proof of cut elimination is detailed. Then we show how a natural deduction formulation can be derived and also how other proposals fail to have certain desirable properties. We consider reduction in the natural deduction formulation, first by eliminating 'detours' in a proof and then by consideration of the subformula property, from which we find the commuting conversions necessary to ensure that this property holds. Next we consider an axiomatic formulation.

We show how all three formulations are equivalent by giving translation procedures between them. Finally we consider how **IL** can be embedded into **ILL** via a translation due to Girard.

- Chapter 3 considers the theory of the linear term calculus. We give the term formation rules as well as the reduction rules which are derived by applying the Curry-Howard correspondence to the reduction rules from the proof theory. Having considered term assignment for the sequent calculus formulation, we reconsider the cut elimination process in the light of the reduction rules it generates on terms. By applying a Curry-Howard-like correspondence to the axiomatic formulation we obtain a linear combinatory logic. We explore reduction within this combinatory formulation. Proofs of strong normalization and confluence are given for a fragment of the reduction rules (the β-reductions). We then consider how linear terms can be translated into linear combinators, before considering and comparing their respective reduction behaviour. Finally we show how terms from the extended λ-calculus can be translated into linear terms and vice versa. We briefly mention some properties of this translation.

- Chapter 4 considers a categorical analysis of **ILL**. We start by viewing the linear term calculus as a (linear) equational logic, which is then analysed to derive a categorical model. After concluding with a notion of a *linear category* we study some of its properties. We then consider two alternative models: firstly one due to Seely and then one due to Lafont. Surprisingly we find that Seely's model does not model all reductions with equal morphisms. In other words it is not sound. We show how after adapting Seely's model using a more abstract setting, it becomes a particular instance of a linear category. We also show how a Lafont model is another particular instance of a linear category. Finally we consider various categories of coalgebras, including some proposed by Hyland. The construction of these categories can be seen as a categorical equivalent to the embedding of **IL** into **ILL**.

- Chapter 5 completes the dissertation by drawing some conclusions and suggesting further work. We consider an alternative natural deduction formulation proposed by Troelstra and in particular some of its categorical consequences.

5 Results

The main contribution of this dissertation is to examine the known relationship between constructive logic, typed λ-calculus and categorical models in the context of **ILL**. Here are some specific original results:

- A proof of cut elimination is given for the sequent calculus formulation of **ILL**.[4]

- A natural deduction formulation is given which is shown to be closed under substitution. Other formulations do not have this fundamental property.

- Reduction in all three systems are compared and contrasted.

- A detailed proof of the subformula property for the natural deduction formulation is given.

- The sequent calculus, natural deduction and axiomatic formulations are shown to be equivalent by giving procedures for mapping proofs in one system to another.

- A proof of Girard's translation function from **IL** to **ILL** is given for the natural deduction formulation.

- A term assignment system is given for **ILL**. Proofs of strong normalization and confluence are given.

- A procedure for 'compiling' linear terms into linear combinators is given. The question of respective reduction behaviour is addressed.

[4]Girard gave a proof based on proof nets for classical linear logic (**CLL**).

- A method for considering the linear term calculus as an equational logic is developed. This results in a general categorical model for **ILL**.
- Seely's model is shown to be unsound. An alternative definition of a Seely-style model is shown to be sound.
- Lafont's model is proved to be sound.
- Proofs are given for Hyland's constructions concerning the category of coalgebras of a linear category.
- A proof is given that an alternative natural deduction formulation of Troelstra, analysed at a categorical level, suggests a model with an idempotent comonad.

6 Prerequisites and Notation

This thesis is intended to be reasonably self-contained. However, the reader is assumed to be familiar with notions of propositional logic and the λ-calculus to the level of a good undergraduate course. Troelstra's recent book on linear logic [75] covers both intuitionistic and classical fragments. The approach to logic taken in this thesis is probably best covered by the book by Girard, Lafont and Taylor [34]. Barendregt's book [8] is essential reading for those interested in the λ-calculus; Hindley and Seldin's book [39] provides a useful alternative view.

The chapter on a categorical model for **ILL** assumes some knowledge of basic category theory. However categorical notions which are particular to the linear set-up of this thesis are defined in the chapter. In accordance with the fact that this is a *computer science* thesis, we shall write function composition in a sequential, left-to-right manner, i.e. $f;g$, rather than the mathematical tradition of $g \circ f$.

On the whole this thesis follows the notation originally used by Girard [31]. However, there are some minor differences with the symbols for the units.

This thesis	Girard
I	**1**
f	**0**
t	$\top$

We have followed Gallier [29] in a systematic use of various symbols to represent differing notions of deduction, rather than a notational abuse of a single turnstile symbol. Thus we use the symbol '$\vdash$' for a turnstile in a logical deduction[5] (i.e. the sequent calculus, natural deduction or axiomatic formulations), '$\triangleright$' for a turnstile in a term assignment system and '$\Rightarrow$' for a turnstile in a combinatory derivation. The '$\vdash$' symbol is used to denote provability (thus we shall write $\vdash \Gamma \vdash A$ rather than the uninformative $\vdash \Gamma \vdash A$). We shall often annotate the '$\vdash$' to qualify the logical system in question when it is not obvious by context.

[5]This symbol is often dubbed a *Girardian* turnstile.

Chapter 2
Proof Theory

1 Sequent Calculus

As explained in Chapter 1, **ILL** arises from removing the structural rules of *Weakening* and *Contraction*. This has the effect of distinguishing between different formulations of the familiar connectives of **IL**. For example, in **IL**, we might formulate the $\wedge_{\mathcal{R}}$ rule in one of two ways.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge'_{\mathcal{R}} \qquad\qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge''_{\mathcal{R}}$$

However, we can see that with rules of *Weakening* and *Contraction* with can simulate one with the other.

$$\frac{\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma, \Gamma \vdash A \wedge B}\,\wedge''_{\mathcal{R}}}{\Gamma \vdash A \wedge B}\,\textit{Contraction}^{*} \qquad \frac{\dfrac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}\,\textit{Weakening}^{*} \qquad \dfrac{\Delta \vdash B}{\Gamma, \Delta \vdash B}\,\textit{Weakening}^{*}}{\Gamma, \Delta \vdash A \wedge B}\,\wedge'_{\mathcal{R}}$$

In **ILL** since we do not have the structural rules these two possible formulations become distinct connectives. We shall use the terminology of Girard to describe these connectives: those where the upper sequent contexts are disjoint (as in ($\wedge''_{\mathcal{R}}$)) are known as the *multiplicatives* and those where the upper sequent contexts must be the same (as in ($\wedge'_{\mathcal{R}}$)) are known as the *additives*. Thus for **ILL** we shall consider the following connectives:

	Connective	Symbol	
Multiplicative	Implication	$\multimap$	"Linear Implication"
	Conjunction	$\otimes$	"Tensor"
Additive	Conjunction	&	"With"
	Disjunction	$\oplus$	"Sum"

It can be seen that there are some obvious omissions from this table, namely multiplicative disjunction and additive implication. Multiplicative disjunction ($\invamp$ or "Par") requires multiple conclusions which is beyond the scope of this thesis. (It was thought that it only made sense as a classical connective, but recent work by Hyland and de Paiva [43] shows how it can be considered as a intuitionistic connective.) The additive implication, $\diamond\!-$, can be formulated as follows.

$$\frac{\Gamma \vdash A \qquad B, \Gamma \vdash C}{\Gamma, A \diamond\!- B \vdash C}\,(\diamond\!-_{\mathcal{L}}) \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \diamond\!- B}\,(\diamond\!-_{\mathcal{R}})$$

However, this connective is generally ignored as its computational content seems minimal.[1] We shall do likewise and not consider this connective further.

We have units for the two conjunctions and the (additive) disjunction. I is the unit for the tensor, **t** is the unit for the With, and **f** is the unit for the Sum.

Of course the logic so far is extremely weak. Girard's innovation was to introduce a new unary connective, !, (the so-called 'exponential') to regain the logical power. The exponential allows a

[1] Troelstra [75, Chapter 4] considers briefly additive implication in the context of a logic with just two implications.

formula to be weakened or multiple occurrences to be contracted. Thus we have the structural rules reinstated, but in a controlled way.

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \textit{Weakening} \qquad\qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \textit{Contraction}$$

However, we need to be able to introduce this new connective, and so we have a rule for introducing it on the left (*Dereliction*) and on the right (*Promotion*).[2] (These are reminiscent of those for the □ connective for modal logics and so are sometimes called the 'modality rules'.)

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \textit{Dereliction} \qquad\qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \textit{Promotion}$$

In the *Promotion* rule, $!\Gamma$ is taken to mean that every formula in the context is of the form $!A_i$. We shall see later, in §5, how introducing this new connective regains the logical power of **IL**. We assume a countably infinite set of propositional symbols (atomic formulae), elements of which we denote with the letters $a, b, \ldots$. A linear formula, A, is then given by the grammar

$$\begin{array}{rcl} A & ::= & a \mid \mathbf{t} \mid \mathbf{f} \mid I \mid \\ & & A \otimes A \mid A \multimap A \mid A \& B \mid A \oplus A \mid !A. \end{array}$$

We give the sequent calculus formulation in Figure 2.1, which is originally due to Girard [33]. Although we have given the *Exchange* rule explicitly, for the rest of this thesis we shall consider this rule to be implicit, whence the convention that Γ, Δ, Θ denote multisets rather than sequences.

Let us state some provable sequents for **ILL**.

Proposition 1. The following are all provable in **ILL**.

1. $A \otimes B \vdash B \otimes A$
2. $A \otimes I \vdash A$ and $A \vdash A \otimes I$
3. $A \& B \vdash B \& A$
4. $A \& \mathbf{t} \vdash A$ and $A \vdash A \& \mathbf{t}$
5. $A \oplus B \vdash B \oplus A$
6. $A \oplus \mathbf{f} \vdash A$ and $A \vdash A \oplus \mathbf{f}$
7. $A \otimes (B \& C) \vdash (A \otimes B) \& (A \otimes C)$
8. $A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)$
9. $(A \otimes B) \oplus (A \otimes C) \vdash A \otimes (B \oplus C)$
10. $!A \otimes !B \vdash !(A \otimes B)$ and $I \vdash !I$
11. $!A \vdash I \& A \& (!A \otimes !A)$
12. $!A \otimes !B \vdash !(A \& B)$ and $!(A \& B) \vdash !A \otimes !B$
13. $I \vdash !\mathbf{t}$ and $!\mathbf{t} \vdash I$

Before considering the sequent calculus formulation in more detail let us fix some standard terminology.

Definition 1.

1. In an instance of a *Cut* rule

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \textit{Cut}$$

the formula A is called the *Cut Formula*.

[2] Girard, Scedrov and Scott [35] call this rule *Storage* in their work on Bounded Linear Logic.

$$\frac{}{A \vdash A}\ \textit{Identity}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \textit{Exchange}$$

$$\frac{\Gamma \vdash B \qquad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \textit{Cut}$$

$$\frac{}{\Gamma \vdash \mathbf{t}}\ (\mathbf{t}_{\mathcal{R}}) \qquad \frac{}{\Gamma, \mathbf{f} \vdash A}\ (\mathbf{f}_{\mathcal{L}})$$

$$\frac{\Gamma \vdash A}{\Gamma, I \vdash A}\ (I_{\mathcal{L}}) \qquad \frac{}{\vdash I}\ (I_{\mathcal{R}})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C}\ (\&_{\mathcal{L}-1}) \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}\ (\&_{\mathcal{L}-2})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{R}-1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{R}-2})$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \textit{Weakening} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \textit{Contraction}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \textit{Dereliction} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \textit{Promotion}$$

Figure 2.1: Sequent Calculus Formulation of **ILL**

2. In a rule the formula being formed is called the *principal formula*, for example in the rule

$$\frac{\Gamma \vdash A \qquad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\,(\multimap_{\mathcal{L}}),$$

$A \multimap B$ is the principal formula, as is $!A$ in the rule

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\textit{Promotion}.$$

3. In a proof π of the form

$$\frac{\dfrac{\vdots}{\Gamma \vdash A}\,R_1 \qquad \dfrac{\vdots}{A, \Delta \vdash B}\,R_2}{\Gamma, \Delta \vdash B}\,\textit{Cut},$$

we shall refer to the application of the *Cut* rule as a (R_1, R_2)-cut.

4. In a proof π of the form

$$\frac{\vdots}{\Gamma \vdash A}\,R_1,$$

we shall write $\mathsf{lst}(\pi)$ to denote the *last rule used* in the proof $(= R_1)$.

5. If a proof π is of the form

$$\frac{\pi_1 \qquad \pi_2}{\Gamma \vdash A}\,R,$$

we shall say that the rule, R, is a *binary rule*. (We shall often refer to the upper proofs as π_1 and π_2.) If a proof π is the form

$$\frac{\pi_1}{\Gamma \vdash A}\,R,$$

we shall say that R is a *unary rule*. (Again, we shall often refer to the upper proof as π_1.)

1.1 Cut Elimination

The *Cut* rule enables us to join two proofs together. Gentzen discovered that applications of the *Cut* rule could be eliminated from a proof. Thus given a proof containing applications of the *Cut* rule, a cut-free version could be found, which can be thought of as its *normal form*. Gentzen's analysis not only showed that the occurrences of the *Cut* rule could be eliminated, but also gave a simple procedure for doing so.

We shall carry out a similar programme for **ILL**. The proof given here is adapted from that for **IL** given by Gallier [29]. First we shall define some measures which we shall use in the proof.

Definition 2.

1. The *rank* of a formula A, is defined as

$$\begin{array}{rcll}
|A| & \stackrel{\text{def}}{=} & 0 & A \text{ is an atomic formula} \\
|I|, |\mathbf{t}|, |\mathbf{f}| & \stackrel{\text{def}}{=} & 0 & \\
|A \multimap B| & \stackrel{\text{def}}{=} & |A| + |B| + 1 & \\
|A \otimes B| & \stackrel{\text{def}}{=} & |A| + |B| + 1 & \\
|A \& B| & \stackrel{\text{def}}{=} & |A| + |B| + 1 & \\
|A \oplus B| & \stackrel{\text{def}}{=} & |A| + |B| + 1 & \\
|!A| & \stackrel{\text{def}}{=} & |A| + 1. &
\end{array}$$

2. The *cut rank* of a proof π, is defined as

$$c(\pi) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{If } \mathsf{lst}(\pi)=\mathit{Identity} \\ c(\pi_1) & \text{If } \mathsf{lst}(\pi)=\text{Unary rule} \\ \max\{c(\pi_1), c(\pi_2)\} & \text{If } \mathsf{lst}(\pi)=\text{Binary rule} \\ \max\{|A|+1, c(\pi_1), c(\pi_2)\} & \text{If } \mathsf{lst}(\pi)=\mathit{Cut} \\ & \text{with Cut formula } A. \end{cases}$$

3. The *depth* of a proof π, is defined as

$$d(\pi) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{If } \mathsf{lst}(\pi)=\mathit{Identity} \\ d(\pi_1)+1 & \text{If } \mathsf{lst}(\pi)=\text{Unary rule} \\ \max\{d(\pi_1), d(\pi_2)\}+1 & \text{If } \mathsf{lst}(\pi)=\text{Binary rule.} \end{cases}$$

To facilitate the proof of cut-elimination[3] we find it convenient to replace the *Cut* rule with an *indexed cut rule*

$$\frac{\Gamma \vdash A \qquad \overbrace{A,\ldots,A}^{n},\Delta \vdash B}{\underbrace{\Gamma,\ldots,\Gamma}_{n},\Delta \vdash B}\;Cut_n.$$

It is clear that this is a derived rule[4] as it represents the sequence of *Cut* rules

$$\frac{\Gamma \vdash A \qquad \dfrac{\dfrac{\Gamma \vdash A \qquad \overbrace{A,A,\ldots,A}^{n},\Delta \vdash B}{\Gamma,A,\ldots,A,\Delta \vdash B}\;Cut}{\begin{array}{c}\vdots\\ \Gamma,\Gamma,\ldots,A,\Delta \vdash B\end{array}}}{\underbrace{\Gamma,\Gamma,\ldots,\Gamma}_{n},\Delta \vdash B}\;Cut.$$

Of course when $n = 1$ this rule is just the familiar *Cut* rule as before. It is clear that if we can prove a cut elimination theorem for this more powerful cut rule, then we have as a corollary the cut elimination theorem for the standard unary rule. (In what follows we shall use the abbreviation A^n to represent the sequence $\underbrace{A,\ldots,A}_{n}$.)

Lemma 1. Let Π_1 be a proof of $\Gamma \vdash A$ and Π_2 be a proof of $\Delta, A^n \vdash B$ and assume that $c(\Pi_1), c(\Pi_2) \leq |A|$. A proof, Π, of $\Gamma^n, \Delta \vdash B$ can be constructed such that $c(\Pi) \leq |A|$.

Proof. We proceed by induction on the sum of the depths of the two proofs, i.e. on $d(\Pi_1) + d(\Pi_2)$, where Π_1 and Π_2 are the immediate subtrees of the proof

$$\frac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\Pi_2\\ A^n,\Delta \vdash B\end{array}}{\Gamma^n,\Delta \vdash B}\;Cut_n.$$

There are numerous cases depending on the structure of Π_1 and Π_2.

1. When the principal formula in the proofs Π_1 and Π_2 is the cut formula, A.

[3] In particular, without this trick of taking a multi-cut rule it seems difficult to find an inductive count that decreases if we keep a 'single' cut rule. It should be possible but, as far as my research has found, it seems to have eluded proof theorists so far.

[4] As it is a derived rule, it is simple to extend the notion of cut rank to handle an occurrence of the Cut_n rule.

(a) $(I_{\mathcal{R}}, I_{\mathcal{L}})$-cut.

$$\dfrac{\dfrac{}{\vdash I}(I_{\mathcal{R}}) \qquad \dfrac{\begin{matrix}\pi_1\\ I^n, \Gamma \vdash B\end{matrix}}{I^{n+1}, \Gamma \vdash B}(I_{\mathcal{L}})}{\Gamma \vdash B} Cut_{n+1}$$

Let Π by the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\dfrac{}{\vdash I}(I_{\mathcal{R}}) \qquad \begin{matrix}\pi_1\\ I^n, \Gamma \vdash B\end{matrix}}{\Gamma \vdash B} Cut_n.$$

By assumption $c(\Pi_1), c(\pi_1) \leq |I|$, then $c(\Pi) \leq |I|$ and we are done.

(b) $(\multimap_{\mathcal{R}}, \multimap_{\mathcal{L}})$-cut.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \multimap B}(\multimap_{\mathcal{R}}) \qquad \dfrac{\begin{matrix}\pi_2\\ (A \multimap B)^n, \Delta \vdash A\end{matrix} \qquad \begin{matrix}\pi_3\\ B, (A \multimap B)^m, \Theta \vdash B\end{matrix}}{(A \multimap B)^{n+m+1}, \Delta, \Theta \vdash C}(\multimap_{\mathcal{L}})}{\Gamma^{n+m+1}, \Delta, \Theta \vdash C} Cut_{n+m+1}$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \multimap B}(\multimap_{\mathcal{R}}) \qquad \begin{matrix}\pi_2\\ (A \multimap B)^n, \Delta \vdash A\end{matrix}}{\Gamma^n, \Delta \vdash A} Cut_n.$$

By assumption that $c(\Pi_1), c(\pi_2) \leq |A \multimap B|$ and hence $c(\Pi') \leq |A \multimap B|$. Let Π'' be the proof

$$\dfrac{\begin{matrix}\Pi'\\ \Gamma^n, \Delta \vdash A\end{matrix} \qquad \begin{matrix}\pi_1\\ \Gamma, A \vdash B\end{matrix}}{\Gamma^{n+1}, \Delta \vdash B} Cut_1.$$

Since we have by assumption that $c(\pi_1) \leq |A \multimap B|$ then by definition $c(\Pi'') \leq \max\{(|A|+1), |A \multimap B|, |A \multimap B|\}$; hence $c(\Pi'') \leq |A \multimap B|$.
Let Π''' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \multimap B}(\multimap_{\mathcal{R}}) \qquad \begin{matrix}\pi_3\\ B, (A \multimap B)^m, \Theta \vdash C\end{matrix}}{B, \Gamma^m, \Theta \vdash C} Cut_m.$$

Since $c(\Pi_1), c(\pi_3) \leq |A \multimap B|$ by assumption, we have that $c(\Pi''') \leq |A \multimap B|$. Finally, we can form the proof, Π,

$$\dfrac{\begin{matrix}\Pi''\\ \Gamma^{n+1}, \Delta \vdash B\end{matrix} \qquad \begin{matrix}\Pi'''\\ B, \Gamma^m, \Theta \vdash C\end{matrix}}{\Gamma^{n+m+1}, \Delta, \Theta \vdash C} Cut_1.$$

Thus by definition $c(\Pi) \leq \max\{(|B|+1), |A \multimap B|, |A \multimap B|\}$; hence $c(\Pi) \leq |A \multimap B|$ and we are done.

(c) $(\otimes_{\mathcal{R}}, \otimes_{\mathcal{L}})$-cut.

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\pi_2}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}(\otimes_{\mathcal{R}}) \qquad \dfrac{\overset{\pi_3}{A, B, (A \otimes B)^n, \Theta \vdash C}}{(A \otimes B)^{n+1}, \Theta \vdash C}(\otimes_{\mathcal{L}})}{\Gamma^{n+1}, \Delta^{n+1}, \Theta \vdash C} Cut_{n+1}$$

Let Π' be the proof obtained by applying the inductive hypothesis to the proof

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\pi_2}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}(\otimes_{\mathcal{R}}) \qquad \overset{\pi_3}{A, B, (A \otimes B)^n, \Theta \vdash C}}{A, B, \Gamma^n, \Delta^n, \Theta \vdash C} Cut_n.$$

We have by assumption $c(\Pi_1), c(\pi_3) \leq |A \otimes B|$ and hence $c(\Pi') \leq |A \otimes B|$. Now let Π'' be the proof

$$\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\Pi'}{A, B, \Gamma^n, \Delta^n, \Theta \vdash C}}{B, \Gamma^{n+1}, \Delta^n, \Theta \vdash C} Cut_1.$$

Since we have by assumption that $c(\pi_1) \leq |A \otimes B|$, then by definition $c(\Pi'') \leq \max\{(|A| + 1), |A \otimes B|, |A \otimes B|\}$. Finally, we can form the following proof, Π,

$$\dfrac{\overset{\pi_2}{\Delta \vdash B} \qquad \overset{\Pi''}{B, \Gamma^{n+1}, \Delta^n, \Theta \vdash C}}{\Gamma^{n+1}, \Delta^{n+1}, \Theta \vdash C} Cut_1.$$

Thus by definition $c(\Pi) \leq \max\{(|B| + 1), |A \otimes B|, |A \otimes B|\}$; hence $c(\Pi) \leq |A \otimes B|$ and we are done.

(d) $(\&_{\mathcal{R}}, \&_{\mathcal{L}-1})$-cut.

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \& B}(\&_{\mathcal{R}}) \qquad \dfrac{\overset{\pi_3}{\Delta, (A \& B)^n, A \vdash C}}{\Delta, (A \& B)^{n+1} \vdash C}\&_{\mathcal{L}-1}}{\Gamma^{n+1}, \Delta \vdash C} Cut_{n+1}$$

Let Π' be the proof obtained by applying the inductive hypothesis to the proof

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \& B}(\&_{\mathcal{R}}) \qquad \overset{\pi_3}{\Delta, (A \& B)^n, A \vdash C}}{\Gamma^n, \Delta \vdash C} Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_3) \leq |A \& B|$ and hence $c(\Pi'|) \leq |A \& B|$. We can form the following proof, Π,

$$\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\Pi'}{\Gamma^n, \Delta \vdash C}}{\Gamma^{n+1}, \Delta \vdash C} Cut_1.$$

Thus by definition $c(\Pi') \leq \max\{|A| + 1, |A \& B|, |A \& B|\}$; hence $c(\Pi) \leq |A \& B|$ and we are done.

(e) $(\&_{\mathcal{R}}, \&_{\mathcal{L}-2})$-cut. Similar to case above.

(f) $(\oplus_{\mathcal{R}-1}, \oplus_{\mathcal{L}})$-cut.

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash A}}{\Gamma \vdash A \oplus B}\oplus_{\mathcal{R}-1} \qquad \dfrac{\overset{\pi_2}{\Delta, (A \oplus B)^n, A \vdash C} \qquad \overset{\pi_3}{\Delta, (A \oplus B)^n, B \vdash C}}{\Delta, (A \oplus B)^{n+1} \vdash C}(\oplus_{\mathcal{L}})}{\Gamma^{n+1}, \Delta \vdash C}\,Cut_{n+1}$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \vdash A}}{\Gamma \vdash A \oplus B}\oplus_{\mathcal{R}-1} \qquad \overset{\pi_2}{\Delta, (A \oplus B)^n, A \vdash C}}{\Gamma^n, \Delta \vdash C}\,Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A \oplus B|$ and hence $c(\Pi') \leq |A \oplus B|$. We can form the following proof, Π,

$$\dfrac{\overset{\pi_1}{\Gamma \vdash A} \qquad \overset{\Pi'}{\Gamma^n, \Delta, A \vdash C}}{\Gamma^{n+1}, \Delta \vdash C}\,Cut_1.$$

Thus by definition $c(\Pi) \leq \max\{|A| + 1, |A \oplus B|, |A \oplus B|\}$; hence $c(\Pi) \leq |A \oplus B|$ and we are done.

(g) $(\oplus_{\mathcal{R}-2}, \oplus_{\mathcal{L}})$-cut. Similar to case above.

(h) (*Promotion, Dereliction*)-cut.

$$\dfrac{\dfrac{\overset{\pi_1}{!\Gamma \vdash A}}{!\Gamma \vdash\, !A}\,Promotion \qquad \dfrac{\overset{\pi_2}{A, !A^n, \Delta \vdash B}}{!A^{n+1}, \Delta \vdash B}\,Dereliction}{!\Gamma^{n+1}, \Delta \vdash B}\,Cut_{n+1}$$

Let Π' be the proof obtained by applying the inductive hypothesis to the proof

$$\dfrac{\dfrac{\overset{\pi_1}{!\Gamma \vdash A}}{!\Gamma \vdash\, !A}\,Promotion \qquad \overset{\pi_2}{A, !A^n, \Delta \vdash B}}{A, !\Gamma^n, \Delta \vdash B}\,Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |!A|$ and hence $c(\Pi') \leq |!A|$. Now let Π be the proof

$$\dfrac{\overset{\pi_1}{!\Gamma \vdash A} \qquad \overset{\Pi'}{A, !\Gamma^n, \Delta \vdash B}}{!\Gamma^{n+1}, \Delta \vdash B}\,Cut_1.$$

Thus by definition $c(\Pi) \leq \max\{|A| + 1, |!A|, |!A|\}$; hence $c(\Pi) \leq |!A|$ and we are done.

(i) (*Promotion, Weakening*)-cut.

$$\dfrac{\dfrac{\overset{\pi_1}{!\Gamma \vdash A}}{!\Gamma \vdash\, !A}\,Promotion \qquad \dfrac{\overset{\pi_2}{!A^n, \Delta \vdash B}}{!A^{n+1}, \Delta \vdash B}\,Weakening}{!\Gamma^{n+1}, \Delta \vdash B}\,Cut_{n+1}$$

Let Π' be the proof obtained by applying the inductive hypothesis to the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ !\Gamma \vdash A\end{matrix}}{!\Gamma \vdash !A}\,Promotion \qquad \begin{matrix}\pi_2\\ !A^n, \Delta \vdash B\end{matrix}}{!\Gamma^n, \Delta \vdash B}\,Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |!A|$ and hence $c(\Pi') \leq |!A|$. Finally take the proof, Π,

$$\dfrac{\begin{matrix}\Pi'\\ !\Gamma^n, \Delta \vdash B\end{matrix}}{!\Gamma^{n+1}, \Delta \vdash B}\,Weakening^*.$$

Hence by definition $c(\Pi) \leq |!A|$ and we are done.

(j) (*Promotion*, *Contraction*)-cut.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ !\Gamma \vdash A\end{matrix}}{!\Gamma \vdash !A}\,Promotion \qquad \dfrac{\begin{matrix}\pi_2\\ !A, !A, !A^n, \Delta \vdash B\end{matrix}}{!A^{n+1}, \Delta \vdash B}\,Contraction}{!\Gamma^{n+1}, \Delta \vdash B}\,Cut_{n+1}$$

Let Π' be the proof obtained by applying the inductive hypothesis to the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ !\Gamma \vdash A\end{matrix}}{!\Gamma \vdash !A}\,Promotion \qquad \begin{matrix}\pi_2\\ !A^{n+2}, \Delta \vdash B\end{matrix}}{!\Gamma^{n+2}, \Delta \vdash B}\,Cut_{n+2}.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |!A|$ and hence $c(\Pi') \leq |!A|$. Finally, let Π be the proof

$$\dfrac{\begin{matrix}\Pi'\\ !\Gamma^{n+2}, \Delta \vdash B\end{matrix}}{!\Gamma^{n+1}, \Delta \vdash B}\,Contraction^*.$$

Hence by definition $c(\Pi) \leq |!A|$ and we are done.

2. When in the proof Π_2 the cut formula, A, is a minor formula. We shall consider each case of the last rule applied in Π_2. The case where the cut formula, A, is a minor formula in proof Π_1 is symmetric (and omitted).

(a) $\mathbf{t}_{\mathcal{R}}$.

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \dfrac{}{A^n, \Delta \vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{R}})}{\Gamma^n, \Delta \vdash \mathbf{t}}\,Cut_n$$

We can form the (cut-free) proof

$$\dfrac{}{\Gamma^n, \Delta \vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{R}}),$$

which has a cut-rank of 0 and so we are done.

(b) $\mathbf{f}_{\mathcal{L}}$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{}{A^n, \Delta, \mathbf{f} \vdash B}\,(\mathbf{f}_{\mathcal{L}})}{\Gamma^n, \Delta, \mathbf{f} \vdash B}\,Cut_n$$

We can form the (cut-free) proof

$$\dfrac{}{\Gamma^n, \Delta, \mathbf{f} \vdash B}\,(\mathbf{f}_{\mathcal{L}}),$$

which has a cut-rank of 0 and so we are done.

(c) $I_{\mathcal{L}}$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ A^n, \Delta \vdash B\end{array}}{A^n, I, \Delta \vdash B}\,(I_{\mathcal{L}})}{\Gamma^n, I, \Delta \vdash B}\,Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ A^n, \Delta \vdash B\end{array}}{\Gamma^n, \Delta \vdash B}\,Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can then form the proof, Π,

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma^n, \Delta \vdash B\end{array}}{\Gamma^n, I, \Delta \vdash B}\,(I_{\mathcal{L}}).$$

Hence by definition $c(\Pi) \leq |A|$ and we are done.

(d) $\otimes_{\mathcal{L}}$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ A^n, B, C, \Delta \vdash D\end{array}}{A^n, B\otimes C, \Delta \vdash D}\,(\otimes_{\mathcal{L}})}{\Gamma^n, B\otimes C, \Delta \vdash D}\,Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ A^n, B, C, \Delta \vdash D\end{array}}{\Gamma^n, B, C, \Delta \vdash D}\,Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can form the proof, Π,

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma^n, B, C, \Delta \vdash D\end{array}}{\Gamma^n, B\otimes C, \Delta \vdash D}\,(\otimes_{\mathcal{L}}).$$

We have by definition that $c(\Pi) \leq |A|$ and we are done.

(e) $\otimes_{\mathcal{R}}$.

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \Delta, A^n \vdash B\end{matrix} \qquad \begin{matrix}\pi_3\\ \Theta, A^m \vdash C\end{matrix}}{\Delta, \Theta, A^{n+m} \vdash B \otimes C}(\otimes_{\mathcal{R}})}{\Gamma^{n+m}, \Delta, \Theta \vdash B \otimes C}Cut_{n+m}$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_2\\ \Delta, A^n \vdash B\end{matrix}}{\Gamma^n, \Delta \vdash B}Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. Let Π'' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_3\\ \Theta, A^m \vdash C\end{matrix}}{\Gamma^m, \Theta \vdash C}Cut_m.$$

We have by assumption that $c(\Pi_1), c(\pi_3) \leq |A|$ and hence $c(\Pi'') \leq |A|$. We can form the proof, Π,

$$\dfrac{\begin{matrix}\Pi'\\ \Gamma^n, \Delta \vdash B\end{matrix} \qquad \begin{matrix}\Pi''\\ \Gamma^m, \Theta \vdash C\end{matrix}}{\Gamma^{n+m}, \Delta, \Theta \vdash B \otimes C}(\otimes_{\mathcal{R}}).$$

We have by definition $c(\Pi) \leq \max\{|A|, |A|\}$; thus $c(\Pi) \leq |A|$ and we are done.

(f) $\multimap_{\mathcal{L}}$.

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \Delta, A^n \vdash B\end{matrix} \qquad \begin{matrix}\pi_3\\ C, A^m, \Theta \vdash D\end{matrix}}{A^{n+m}, B \multimap C, \Delta, \Theta \vdash D}(\multimap_{\mathcal{L}})}{\Gamma^{n+m}, B \multimap C, \Delta, \Theta \vdash D}Cut_{n+m}$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_2\\ \Delta, A^n \vdash B\end{matrix}}{\Gamma^n, \Delta \vdash B}Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. Let Π'' be the proof obtained by applying the inductive hypothesis to the proof

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_3\\ C, A^m, \Theta \vdash D\end{matrix}}{C, \Gamma^m, \Theta \vdash D}Cut_m.$$

We have by assumption that $c(\Pi_1), c(\pi_3) \leq |A|$ and hence $c(\Pi'') \leq |A|$. We can form the proof, Π,

$$\dfrac{\begin{matrix}\Pi'\\ \Gamma^n, \Delta \vdash B\end{matrix} \qquad \begin{matrix}\Pi''\\ C, \Gamma^m, \Theta \vdash D\end{matrix}}{\Gamma^{n+m}, \Delta, B \multimap C, \Theta \vdash D}(\multimap_{\mathcal{L}}).$$

We have by definition $c(\Pi) \leq \max\{|A|, |A|\}$; thus $c(\Pi) \leq |A|$ and we are done.

(g) $\multimap_{\mathcal{R}}$.

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ A^n, \Delta, B \vdash C\end{matrix}}{A^n, \Delta \vdash B \multimap C}(\multimap_{\mathcal{R}})}{\Gamma^n, \Delta \vdash B \multimap C}\, Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_2\\ A^n, \Delta, B \vdash C\end{matrix}}{\Gamma^n, \Delta, B \vdash C}\, Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can then form the proof, Π,

$$\dfrac{\begin{matrix}\Pi'\\ \Gamma^n, \Delta, B \vdash C\end{matrix}}{\Gamma^n, \Delta \vdash B \multimap C}\,(\multimap_{\mathcal{L}}).$$

Hence by definition $c(\Pi) \leq |A|$ and we are done.

(h) $\&_{\mathcal{L}-1}$.

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \Delta, A^n, B \vdash D\end{matrix}}{\Delta, A^n, B\&C \vdash D}}{\Gamma^n, \Delta, B\&C \vdash D}\, Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_2\\ \Delta, A^n, B \vdash D\end{matrix}}{\Gamma^n, \Delta, B \vdash D}\, Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can form the proof, Π,

$$\dfrac{\begin{matrix}\Pi'\\ \Gamma^n, \Delta, B \vdash D\end{matrix}}{\Gamma^n, \Delta, B\&C \vdash D}\,\&_{\mathcal{L}-1}.$$

Hence by definition $c(\Pi) \leq |A|$ and we are done.

(i) $\&_{\mathcal{L}-2}$. Similar to case above.

(j) $\&_{\mathcal{R}}$.

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \Delta, A^n \vdash B\end{matrix} \qquad \begin{matrix}\pi_3\\ \Delta, A^n \vdash C\end{matrix}}{\Delta, A^n \vdash B\&C}(\&_{\mathcal{R}})}{\Gamma^n, \Delta \vdash B\&C}\, Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{matrix}\Pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_2\\ \Delta, A^n \vdash B\end{matrix}}{\Gamma^n, \Delta \vdash B}\, Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. Let Π'' be the proof obtained by applying the induction hypothesis to the proof

$$\frac{\overset{\Pi_1}{\Gamma \vdash A} \qquad \overset{\pi_3}{\Delta, A^n \vdash C}}{\Gamma^n, \Delta \vdash C}\ Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_3) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can form the proof, Π,

$$\frac{\overset{\Pi'}{\Gamma^n, \Delta \vdash B} \qquad \overset{\Pi''}{\Gamma^n, \Delta \vdash C}}{\Gamma^n, \Delta \vdash B \& C}\ (\&_{\mathcal{R}}).$$

Hence by definition $c(\Pi) \leq |A|$ and we are done.

(k) $\oplus_{\mathcal{L}}$.

$$\frac{\overset{\Pi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\pi_2}{\Delta, A^n, B \vdash D} \qquad \overset{\pi_3}{\Delta, A^n, C \vdash D}}{\Delta, A^n, B \oplus C \vdash D}\ (\oplus_{\mathcal{L}})}{\Gamma^n, \Delta, B \oplus C \vdash D}\ Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\frac{\overset{\Pi_1}{\Gamma \vdash A} \qquad \overset{\pi_2}{\Delta, A^n, B \vdash D}}{\Gamma^n, \Delta, B \vdash D}\ Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. Let Π'' be the proof obtained by applying the induction hypothesis to the proof

$$\frac{\overset{\Pi_1}{\Gamma \vdash A} \qquad \overset{\pi_3}{\Delta, A^n, C \vdash D}}{\Gamma^n, \Delta, C \vdash D}\ Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_3) \leq |A|$ and hence $c(\Pi'') \leq |A|$. We can form the proof, Π,

$$\frac{\overset{\Pi'}{\Gamma^n, \Delta, B \vdash D} \qquad \overset{\Pi''}{\Gamma^n, \Delta, C \vdash D}}{\Gamma^n, \Delta, B \oplus C \vdash D}\ (\oplus_{\mathcal{L}}).$$

Hence by definition $c(\Pi) \leq |A|$ and we are done.

(l) $\oplus_{\mathcal{R}-1}$.

$$\frac{\overset{\Pi_1}{\Gamma \vdash A} \qquad \dfrac{\overset{\pi_2}{\Delta, A^n \vdash B}}{\Delta, A^n \vdash B \oplus C}}{\Gamma^n, \Delta \vdash B \oplus C}\ Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \Delta, A^n \vdash B\end{array}}{\Gamma^n, \Delta \vdash B}\; Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can then form the proof, Π,

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma^n, \Delta \vdash B\end{array}}{\Gamma^n, \Delta \vdash B \oplus C}\; \oplus_{\mathcal{R}-1}.$$

Hence by definition $c(\Pi) \leq |A|$ and we are done.

(m) $\oplus_{\mathcal{R}-2}$. Similar to case above.

(n) *Dereliction.*

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ A^n, B, \Delta \vdash C\end{array}}{A^n, !B, \Delta \vdash C}\; Dereliction}{\Gamma^n, !B, \Delta \vdash C}\; Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ A^n, B, \Delta \vdash C\end{array}}{\Gamma^n, B, \Delta \vdash C}\; Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can form the proof

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma^n, B, \Delta \vdash C\end{array}}{\Gamma^n, !B, \Delta \vdash C}\; Dereliction.$$

We have by definition that $c(\Pi) \leq |A|$ and we are done.

(o) *Weakening.*

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ A^n, \Delta \vdash C\end{array}}{A^n, !B, \Delta \vdash C}\; Weakening}{\Gamma^n, !B, \Delta \vdash C}\; Cut_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ A^n, \Delta \vdash C\end{array}}{\Gamma^n, \Delta \vdash C}\; Cut_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can then form the proof, Π,

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma^n, \Delta \vdash C\end{array}}{\Gamma^n, !B, \Delta \vdash C}\; Weakening.$$

By definition $c(\Pi) \leq |A|$ and we are done.

(p) *Contraction.*

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ A^n, !B, !B, \Delta \vdash C\end{array}}{A^n, !B, \Delta \vdash C}\ \textit{Contraction}}{\Gamma^n, !B, \Delta \vdash C}\ \textit{Cut}_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ A^n, !B, !B, \Delta \vdash C\end{array}}{\Gamma^n, !B, !B, \Delta \vdash C}\ \textit{Cut}_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |A|$ and hence $c(\Pi') \leq |A|$. We can form the proof, Π,

$$\dfrac{\begin{array}{c}\Pi'\\ \Gamma^n, !B, !B, \Delta \vdash C\end{array}}{\Gamma^n, !B, \Delta \vdash C}\ \textit{Contraction.}$$

By definition $c(\Pi) \leq |A|$ and we are done.

(q) *Promotion.*

$$\dfrac{\begin{array}{c}\Pi_1\\ !\Gamma \vdash !A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ !A^n, !\Delta \vdash B\end{array}}{!A^n, !\Delta \vdash !B}\ \textit{Promotion}}{!\Gamma^n, !\Delta \vdash !B}\ \textit{Cut}_n$$

Let Π' be the proof obtained by applying the induction hypothesis to the proof

$$\dfrac{\begin{array}{c}\Pi_1\\ !\Gamma \vdash !A\end{array} \qquad \begin{array}{c}\pi_2\\ !A^n, !\Delta \vdash B\end{array}}{!\Gamma^n, !\Delta \vdash B}\ \textit{Cut}_n.$$

We have by assumption that $c(\Pi_1), c(\pi_2) \leq |!A|$ and hence $c(\Pi') \leq |!A|$. We can form the proof, Π,

$$\dfrac{\begin{array}{c}\Pi'\\ !\Gamma^n, !\Delta \vdash B\end{array}}{!\Gamma^n, !\Delta \vdash !B}\ \textit{Promotion.}$$

By definition $c(\Pi) \leq |!A|$ and we are done.

3. When either Π_1 or Π_2 is an instance of the *Identity* rule. Firstly

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{}{A \vdash A}\ \textit{Identity}}{\Gamma \vdash A}\ \textit{Cut,}$$

which is replaced by

$$\begin{array}{c}\Pi_1\\ \Gamma \vdash A.\end{array}$$

And similarly the proof

$$\dfrac{\dfrac{}{A \vdash A}\,Identity \qquad \begin{array}{c}\Pi_2\\ \Gamma, A \vdash B\end{array}}{\Gamma, A \vdash B}\,Cut_1,$$

is replaced by

$$\begin{array}{c}\Pi_2\\ \Gamma, A \vdash B.\end{array}$$

In both cases the assumption that $c(\Pi_1), c(\Pi_2) \leq |A|$ ensures that $c(\Pi) \leq |A|$.

■

Lemma 2. Let π be a proof, $\Gamma \vdash A$, with cut rank $c(\pi)$. If $c(\pi) > 0$ then we can construct a proof π' of $\Gamma \vdash A$ such that $c(\pi') < c(\pi)$.

Proof. By inspection of the last rule of proof π. If it is not an instance of the *Cut* rule then we simply apply induction on the subproofs of π. If the last rule in π is a *Cut* then π is of the form

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \Delta, A^n \vdash B\end{array}}{\Gamma^n, \Delta \vdash B}\,Cut_n.$$

If $c(\pi) > |A|+1$ then we can apply induction on the subproofs π_1, π_2 and we are done. If $c(\pi) = |A|+1$ then we can apply Lemma 1 to get a proof π' where $c(\pi') \leq |A|$; hence $c(\pi') < |A| + 1 = c(\pi)$ and we are done. ■

Theorem 1. Let π be a proof of $\Gamma \vdash A$ with cut rank $c(\pi)$. A cut free proof π' of $\Gamma \vdash A$ can be constructed.

Proof. By induction on $c(\pi)$ and Lemma 2. ■

1.2 Cut Elimination and The Additive Units

It seems appropriate to mention here a small problem with the process of cut elimination and the additive units. Consider the following proof which contains one instance of the *Cut* rule.

$$\dfrac{\dfrac{}{\Gamma, \mathbf{f} \vdash A}\,(\mathbf{f}_{\mathcal{L}}) \qquad \dfrac{}{A, \Delta \vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{R}})}{\Gamma, \Delta, \mathbf{f} \vdash \mathbf{t}}\,Cut$$

The problem is to decide which axiom it is replaced by: is it $\mathbf{f}_{\mathcal{L}}$ or $\mathbf{t}_{\mathcal{R}}$? Of course, as far as proving the cut elimination theorem it does not matter which is chosen, but there is little to guide us either way (even the model theoretic viewpoint of the Chapter 4 says little). In the original treatise [31, Page 67], Girard rewrites the above to an instance of the $\mathbf{f}_{\mathcal{L}}$ rule. However, in personal communication, Girard admits to this being a choice made for "...aesthetic reasons". A more convincing explanation of the proof theoretic rôle of the additive units remains an open problem.

1.3 Subformula Property

An immediate consequence of the cut elimination theorem is the subformula property, which follows from the simple observation that every rule except the *Cut* rule has the property that the premises are made up of subformulae of the conclusion.[5]

Definition 3. The *subformulae* of a formula A are defined by the following clauses.

- If A is atomic then its subformula is A.
- The subformulae of $A \otimes B$ are the subformulae of A and the subformulae of B and $A \otimes B$ itself.
- The subformulae of $A \multimap B$ are the subformulae of A and the subformulae of B and $A \multimap B$ itself.
- The subformulae of $A \& B$ are the subformulae of A and the subformulae of B and $A \& B$ itself.
- The subformulae of $A \oplus B$ are the subformulae of A and the subformulae of B and $A \oplus B$ itself.
- The subformulae of $!A$ are the subformulae of A and $!A$ itself.

Theorem 2. In a cut-free proof of $\Gamma \vdash A$ all the formulae which occur within it are contained in the set of subformulae of Γ and A.

Proof. By a simple induction on the structure of the proof $\Gamma \vdash A$. ∎

There are many other consequences to having a cut elimination theorem for a given logic and these have been studied by Schwichtenberg [67]. An example is Craig's Interpolation Lemma which has been studied for some fragments of (classical) linear logic by Roorda [64]. We leave further investigation of this and other properties for future work.

2 Natural Deduction

As explained in Chapter 1, in a natural deduction formulation a deduction is a derivation of a proposition from a finite set of assumption packets using some predefined set of inference rules. More specifically, these packets consist of a multiset of propositions, which may be empty. This flexibility is the equivalent of the Weakening and Contraction rules in the sequent calculus. Within a deduction, we may 'discharge' any number of assumption packets. Assumption packets can be given natural number labels and applications of inference rules can be annotated with the labels of those packets which it discharges.

We might then ask what restrictions need we make to natural deduction to make it linear? Clearly, we need to withdraw the concept of packets of assumptions. A packet must contain exactly one proposition, i.e. a packet is now equivalent to a proposition. A rule which previously discharged many packets of the same proposition, can now only discharge the one. Thus we can label every proposition with a *unique* label.

Before considering the rules, we shall fix some (standard) notation.

Definition 4.

1. In a rule of the form

$$\frac{\begin{matrix}\vdots & & \vdots \\ A_1 & \dots & A_n\end{matrix}}{B},$$

   the A_i are known as *premises* and B as the *conclusion*.
2. In an elimination rule, the premise being eliminated is known as the *major premise*. Any other premises are known as *minor premises*.
3. An assumption which has not been discharged will often be referred to as an *open* assumption.

[5] We shall prove a similar theorem for the natural deduction formulation in the next section.

The Multiplicatives

The introduction rule for linear implication, $\multimap$, is as usual for **IL**; the restriction on packets means that we can only discharge a single assumption. The elimination rule for $\multimap$ is also as for **IL**. The rules are

$$\frac{\begin{array}{c}[A^x]\\ \vdots \\ B\end{array}}{A\multimap B}\,(\multimap_{\mathcal{I}})_x, \quad\text{and}\quad \frac{\begin{array}{c}\vdots\\ A\multimap B\end{array}\qquad \begin{array}{c}\vdots\\ A\end{array}}{B}\,(\multimap_{\mathcal{E}}).$$

We should note that in the elimination rule for $\multimap$ the multiplicative nature is implicit in our restriction that all assumptions have unique labels. Hence to bring two derivations together implies that their contexts are disjoint.

The introduction rule for tensor, $\otimes$, is as usual for **IL**; with again the implicit multiplicative nature. The elimination rule is slightly more surprising. In **IL** we are used to having two elimination rules for the conjunction which enable one of the conjuncts to be 'projected' out of a conjunction, *viz.*

$$\frac{A\wedge B}{A}\,(\wedge_{\mathcal{E}-1}), \quad\text{and}\quad \frac{A\wedge B}{B}\,(\wedge_{\mathcal{E}-2}).$$

However **ILL** does not permit projection over multiplicative conjunction (as it would provide unrestricted *Weakening*), but rather both components of the tensor should be used in the deduction. Thus the elimination rule is of the form[6]

$$\frac{\begin{array}{c}\vdots\\ A\otimes B\end{array}\qquad \begin{array}{c}[A^x]\ [B^y]\\ \vdots\\ C\end{array}}{C}\,(\otimes_{\mathcal{E}})_{x,y}.$$

The Exponential

The elimination rule, *Dereliction*, is easy to formulate, *viz.*

$$\frac{\begin{array}{c}\vdots\\ !B\end{array}}{B}\,\mathit{Dereliction}.$$

However it is surprising to note that other presentations [57, 55, 79] have adopted the following slightly more verbose formulation

$$\frac{\begin{array}{c}\vdots\\ !B\end{array}\qquad \begin{array}{c}[B^x]\\ \vdots\\ C\end{array}}{C}\,\mathit{Dereliction}'_x.$$

The *Weakening* rule allows a deduction whose conclusion is of the form $!B$ to play no part in another deduction, or in other words, it allows us to build dummy deductions. Again, the intuitive formulation is

$$\frac{\begin{array}{c}\vdots\\ !B\end{array}\qquad \begin{array}{c}\vdots\\ C\end{array}}{C}\,\mathit{Weakening}.$$

[6]Schroeder-Heister [66] has considered natural deduction formulations of connectives of this form (although for **IL**).

The *Contraction* rule allows the result of a deduction to be used twice as an assumption. This rule is realized in **IL** by the implicit ability to give two assumptions the same label. We can then substitute a deduction for this duplicated assumption by duplicating the deduction. Duplicating a deduction is illegal in our linear system because we cannot have duplicated labels. We must formulate the rule so that the deduction appears once and its conclusion appears twice with different labels, *viz.*

$$\dfrac{\begin{array}{c}\vdots\\ !B\end{array} \qquad \begin{array}{c}[!B^x]\ [!B^y]\\ \vdots\\ C\end{array}}{C}\ \textit{Contraction}_{x,y}.$$

Now we come to the problematic rule of *Promotion*. This rule insists that all the assumptions at the time of application are of the form $!A_i$. Thus as a first attempt at a formulation we shall take the following (as have all other proposals [7, 57, 55, 79, 75])

$$\dfrac{\begin{array}{c}!A_1\ \cdots\ !A_n\\ \vdots\\ B\end{array}}{!B}\ \textit{Promotion.} \tag{2.1}$$

Let us consider a fundamental feature of any natural deduction formulation.

Definition 5. A natural deduction formulation is said to be *closed under substitution* if the following is satisfied: if for any two valid deductions

$$\begin{array}{c}\Gamma\\ \vdots\\ A,\end{array} \qquad \text{and} \qquad \begin{array}{c}A\quad \Delta\\ \vdots\\ B,\end{array}$$

the following is a valid deduction

$$\begin{array}{c}\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\qquad \Delta\\ \vdots\\ B.\end{array}$$

It is quite clear that the formulation for the *Promotion* rule given above is *not* closed under substitution. For example consider substituting the deduction

$$\dfrac{C \multimap !A_1 \qquad C}{!A_1}\ (\multimap_{\mathcal{E}}),$$

for the assumption $!A_1$ in 2.1. We would arrive at the deduction

$$\dfrac{\begin{array}{c}\dfrac{C \multimap !A_1 \qquad C}{!A_1}\ (\multimap_{\mathcal{E}})\quad \cdots\ !A_n\\ \vdots\\ B\end{array}}{!B}\ \textit{Promotion.}$$

This deduction is *not valid*, as the assumptions are not all of the form $!A_i$. To gain a correct formulation we need to make the substitutions explicit. In this thesis we shall use the following formulation.

$$\frac{\begin{array}{c}\vdots\\ !A_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ !A_n\end{array} \qquad \begin{array}{c}[\![!A_1^{x_1}\ \cdots\ !A_n^{x_n}]\!]\\ \vdots\\ B\end{array}}{!B}\ \mathit{Promotion}_{x1,\ldots,xn}$$

Some care needs to be taken with this rule. The semantic brackets $[\![\ldots]\!]$ signify that to apply this rule correctly, not only must *all* the assumptions be of the form $!A_i$, but that they are *all* discharged and re-introduced. It is obvious that this formulation will be closed under substitution. We shall use the terminology that the $!A_i$ are called the *minor* premises of this rule.

The Additives

The additives seem to conflict with the notion of linearity which we have described so far. We have seen that in the sequent calculus formulation they require the upper sequent contexts to be equal, for example in the $(\&_\mathcal{R})$ rule

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A\&B}\ (\&_\mathcal{R}).$$

In what we have done so far, contexts are implicitly disjoint, so it is clear that we have to get round this restriction. Taking as an example the introduction rule for the With connective we can isolate two distinct proposals.

1. Keep the implicit disjointness of contexts. Thus when the additive contexts are brought together, we have to immediately discharge them both and reintroduce them once. Thus the rule would be

$$\frac{\begin{array}{c}\vdots\\ A_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ A_n\end{array} \qquad \begin{array}{c}[\![A_1^{x_1}\ \cdots\ A_n^{x_n}]\!]\\ \vdots\\ B\end{array} \qquad \begin{array}{c}[\![A_1^{y_1}\ \cdots\ A_n^{y_n}]\!]\\ \vdots\\ C\end{array}}{B\&C}\ (\&_\mathcal{I})_{x1,\ldots,xn,y1,\ldots,yn}.$$

   This rule has the restriction that the two (discharged) contexts are equal. It should be noted that this equality of the contexts only applies to the assumptions, *not* their labels.[7]

2. Extend the proof theory, so as to introduce seriously the notion of an *additive context*. This extension allows us to bring together equal contexts and treat them as a single context. Thus the $(\&_\mathcal{I})$ rule would be

$$\begin{array}{c} \Gamma \\ \swarrow \quad \searrow \\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad \begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A\&B}\ (\&_\mathcal{I}).\end{array}$$

   It should be noted that the labels in the contexts are required to be the same.

In some senses the second proposal can be seen as an implementation method for the first. Certainly there does not appear to be any fundamental problem with either proposal (unlike the situation with the *Promotion* rule). We then have a choice, and for this thesis we shall take the second

[7] It should also be noted that this formulation corresponds to the additive boxes used by Girard [31] for the proof net formulation of Classical Linear Logic.

proposal. This proposal has the advantage of better proof-theoretic properties as well as a more succinct syntax.

The elimination rules for With are simply the familiar projection rules from **IL**,

$$\dfrac{\begin{array}{c}\vdots\\ A\&B\end{array}}{A}\,\&_{\mathcal{E}-1}, \qquad \text{and} \qquad \dfrac{\begin{array}{c}\vdots\\ A\&B\end{array}}{B}\,\&_{\mathcal{E}-2}.$$

Thus the additive conjunction is like an external choice (as either conjunct can be extracted). The introduction rules for additive disjunction are as for **IL**,

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A\oplus B}\,\oplus_{\mathcal{I}-1}, \qquad \text{and} \qquad \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A\oplus B}\,\oplus_{\mathcal{I}-2}.$$

The elimination rule for the additive disjunction is also as for **IL**,

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\oplus B\end{array} \qquad \begin{array}{c}[A^x]\ \ \Delta\\ \vdots\\ C\end{array} \qquad \begin{array}{c}\Delta\ \ [B^y]\\ \vdots\\ C\end{array}}{C}\,(\oplus_{\mathcal{E}})_{x,y}.$$

Thus the additive disjunction is like an internal choice (as the disjunct itself determines whether it is an A or a B and to deal with it we need to provide a case for both possibilities).

The Additive Units

The additive units are simply the nullary formulations of the additive rules. Thus from the formulation of the additives we can derive the formulation of the units. Firstly, the $\mathbf{t}_{\mathcal{I}}$ rule which is the nullary version of the $\&_{\mathcal{I}}$ rule,

$$\dfrac{\begin{array}{c}\vdots\\ A_1\end{array}\ \cdots\ \begin{array}{c}\vdots\\ A_n\end{array}}{\mathbf{t}}\,(\mathbf{t}_{\mathcal{I}}).$$

Next the $\mathbf{f}_{\mathcal{E}}$ rule, which is the nullary version of the $\oplus_{\mathcal{E}}$ rule

$$\dfrac{\begin{array}{c}\vdots\\ A_1\end{array}\ \cdots\ \begin{array}{c}\vdots\\ A_n\end{array}\quad \begin{array}{c}\vdots\\ \mathbf{f}\end{array}}{B}\,(\mathbf{f}_{\mathcal{E}}).$$

It should be noted that another formulation of the $(\mathbf{f}_{\mathcal{E}})$ rule, namely

$$\dfrac{\begin{array}{c}\vdots\\ \mathbf{f}\end{array}}{A}\,(\mathbf{f}_{\mathcal{E}})',$$

would be an acceptable alternative (in that it is of equal expressive power).

For completeness we shall repeat the entire natural deduction formulation in Figure 2.2. The advantage of this formulation over others is that it has the following property.

Theorem 3. The natural deduction formulation is closed under substitution.

$$\frac{\begin{array}{c}[A^x]\\ \vdots\\ B\end{array}}{A \multimap B}\,(\multimap_{\mathcal{I}})_x \qquad \frac{\begin{array}{c}\vdots\\ A\multimap B\end{array} \quad \begin{array}{c}\vdots\\ A\end{array}}{B}\,(\multimap_{\mathcal{E}})$$

$$\frac{\begin{array}{c}\vdots\\ A_1\end{array} \;\cdots\; \begin{array}{c}\vdots\\ A_n\end{array}}{\mathbf{t}}\,(\mathbf{t}_{\mathcal{I}}) \qquad \frac{\begin{array}{c}\vdots\\ A_1\end{array} \;\cdots\; \begin{array}{c}\vdots\\ A_n\end{array} \quad \begin{array}{c}\vdots\\ \mathbf{f}\end{array}}{B}\,(\mathbf{f}_{\mathcal{E}})$$

$$\frac{}{I}\,(I_{\mathcal{I}}) \qquad \frac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ I\end{array}}{A}\,(I_{\mathcal{E}})$$

$$\frac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ B\end{array}}{A\otimes B}\,(\otimes_{\mathcal{I}}) \qquad \frac{\begin{array}{c}\vdots\\ A\otimes B\end{array} \quad \begin{array}{c}[A^x]\;[B^y]\\ \vdots\\ C\end{array}}{C}\,(\otimes_{\mathcal{E}})_{x,y}$$

$$\Gamma \qquad\qquad\qquad\qquad \Delta$$

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \quad \begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A\& B}\,(\&_{\mathcal{I}}) \qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ A\oplus B\end{array} \quad \begin{array}{c}[A^x]\;\Delta\\ \vdots\\ C\end{array} \quad \begin{array}{c}\Delta\;[B^y]\\ \vdots\\ C\end{array}}{C}\,(\oplus_{\mathcal{E}})_{x,y}$$

$$\frac{\begin{array}{c}\vdots\\ A\& B\end{array}}{A}\,\&_{\mathcal{E}-1} \quad \frac{\begin{array}{c}\vdots\\ A\& B\end{array}}{B}\,\&_{\mathcal{E}-2} \qquad \frac{\begin{array}{c}\vdots\\ A\end{array}}{A\oplus B}\,\oplus_{\mathcal{I}-1} \quad \frac{\begin{array}{c}\vdots\\ B\end{array}}{A\oplus B}\,\oplus_{\mathcal{I}-2}$$

$$\frac{\begin{array}{c}\vdots\\ !B\end{array} \quad \begin{array}{c}\vdots\\ C\end{array}}{C}\,\textit{Weakening} \qquad \frac{\begin{array}{c}\vdots\\ !B\end{array} \quad \begin{array}{c}[!B^x]\;[!B^y]\\ \vdots\\ C\end{array}}{C}\,\textit{Contraction}_{x,y}$$

$$\frac{\begin{array}{c}\vdots\\ !B\end{array}}{B}\,\textit{Dereliction} \qquad \frac{\begin{array}{c}\vdots\\ !A_1\end{array} \;\ldots\; \begin{array}{c}\vdots\\ !A_n\end{array} \quad \begin{array}{c}[\![!A_1^{x_1} \;\cdots\; !A_n^{x_n}]\!]\\ \vdots\\ B\end{array}}{!B}\,\textit{Promotion}_{x1,\ldots,xn}$$

Figure 2.2: Natural Deduction Formulation of **ILL**

It is possible to present natural deduction rules in a 'sequent-style', where given a sequent $\Gamma \vdash A$, Γ represents all the undischarged assumptions so far in the deduction, and A represents conclusion of the deduction. We can still label the undischarged assumptions with a unique natural number, but we refrain from doing so. This formulation should not be confused with the sequent calculus formulation given in §1, which differs by having operations which act on the left and right of the turnstile, rather than rules for the introduction and elimination of logical connectives. The 'sequent-style' formulation of natural deduction is given in Figure 2.3.

2.1 β-Reductions

With a natural deduction formulation we can produce so-called 'detours' in a deduction, which arise where we introduce a logical connective and then eliminate it immediately afterwards. We can define the normalization procedure by considering each case of an introduction rule followed immediately by a corresponding elimination rule in turn.

- $(\multimap_\mathcal{I})$ followed by $(\multimap_\mathcal{E})$

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \multimap B}(\multimap_\mathcal{I}) \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B}(\multimap_\mathcal{E})$$

normalizes to

$$\begin{matrix}\vdots\\ [A]\\ \vdots\\ B.\end{matrix}$$

- $(I_\mathcal{I})$ followed by $(I_\mathcal{E})$

$$\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \dfrac{}{I}(I_\mathcal{I})}{A}(I_\mathcal{E})$$

normalizes to

$$\begin{matrix}\vdots\\ A.\end{matrix}$$

- $(\otimes_\mathcal{I})$ followed by $(\otimes_\mathcal{E})$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ B\end{matrix}}{A \otimes B}(\otimes_\mathcal{I}) \qquad \begin{matrix}[A]\ [B]\\ \vdots\\ C\end{matrix}}{C}(\otimes_\mathcal{E})$$

normalizes to

$$\begin{matrix}\vdots \quad \vdots\\ [A]\ \ [B]\\ \vdots\\ C.\end{matrix}$$

$$\frac{}{A \vdash A}\ \textit{Identity}$$

$$\frac{\Gamma_1 \vdash A_1 \quad \cdots \quad \Gamma_n \vdash A_n}{\Gamma_1, \ldots, \Gamma_n \vdash \mathbf{t}}\ (\mathbf{t}_{\mathcal{I}}) \qquad \frac{\Gamma_1 \vdash A_1 \quad \cdots \quad \Gamma_n \vdash A_n \qquad \Delta \vdash \mathbf{f}}{\Gamma_1, \ldots, \Gamma_n, \Delta \vdash A}\ (\mathbf{f}_{\mathcal{E}})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{I}}) \qquad \frac{\Gamma \vdash A \multimap B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ (\multimap_{\mathcal{E}})$$

$$\frac{}{\vdash I}\ (I_{\mathcal{I}}) \qquad \frac{\Delta \vdash I \qquad \Gamma \vdash A}{\Gamma, \Delta \vdash A}\ (I_{\mathcal{E}})$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes_{\mathcal{I}}) \qquad \frac{\Gamma \vdash A \otimes B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}\ (\otimes_{\mathcal{E}})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&_{\mathcal{I}}) \qquad \frac{\Gamma \vdash A \& B}{\Gamma \vdash A}\ \&_{\mathcal{E}-1} \qquad \frac{\Gamma \vdash A \& B}{\Gamma \vdash B}\ (\&_{\mathcal{E}-2})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{I}-1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{I}-2})$$

$$\frac{\Gamma \vdash A \oplus B \qquad \Delta, A \vdash C \qquad \Delta, B \vdash C}{\Gamma, \Delta \vdash C}\ (\oplus_{\mathcal{E}})$$

$$\frac{\Delta_1 \vdash !A_1 \quad \cdots \quad \Delta_n \vdash !A_n \qquad !A_1, \ldots, !A_n \vdash B}{\Delta_1, \ldots, \Delta_n \vdash !B}\ \textit{Promotion}$$

$$\frac{\Gamma \vdash !A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \textit{Weakening} \qquad \frac{\Gamma \vdash !A \qquad \Delta, !A, !A \vdash B}{\Gamma, \Delta \vdash B}\ \textit{Contraction}$$

$$\frac{\Gamma \vdash !A}{\Gamma \vdash A}\ \textit{Dereliction}$$

Figure 2.3: Natural Deduction Formulation of ILL in Sequent-Style

- $(\&_{\mathcal{I}})$ followed by $(\&_{\mathcal{E}-1})$.

$$\begin{array}{c}\Gamma\\ \swarrow\quad\searrow\end{array}$$

$$\cfrac{\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad \begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A\& B}\,(\&_{\mathcal{I}})}{A}\,(\&_{\mathcal{E}-1})$$

normalizes to

$$\begin{array}{c}\Gamma\\ \vdots\\ A.\end{array}$$

- $(\&_{\mathcal{I}})$ followed by $(\&_{\mathcal{E}-2})$.

$$\begin{array}{c}\Gamma\\ \swarrow\quad\searrow\end{array}$$

$$\cfrac{\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad \begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A\& B}\,(\&_{\mathcal{I}})}{B}\,(\&_{\mathcal{E}-2})$$

normalizes to

$$\begin{array}{c}\Gamma\\ \vdots\\ B.\end{array}$$

- $(\oplus_{\mathcal{I}-1})$ followed by $(\oplus_{\mathcal{E}})$.

$$\begin{array}{c}\Delta\\ \swarrow\quad\searrow\end{array}$$

$$\cfrac{\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A\oplus B}\,(\oplus_{\mathcal{I}-1}) \qquad \begin{array}{c}[A^x]\ \ \Delta\\ \vdots\\ C\end{array} \qquad \begin{array}{c}\Delta\ \ [B^y]\\ \vdots\\ C\end{array}}{C}\,(\oplus_{\mathcal{E}})_{x,y}$$

normalizes to

$$\begin{array}{c}\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\quad \Delta\\ \vdots\\ C.\end{array}$$

- $(\oplus_{\mathcal{I}-2})$ followed by $(\oplus_{\mathcal{E}})$.

$$\begin{array}{c}\Delta\\ \swarrow\quad\searrow\end{array}$$

$$\cfrac{\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A\oplus B}\,(\oplus_{\mathcal{I}-2}) \qquad \begin{array}{c}[A^x]\ \ \Delta\\ \vdots\\ C\end{array} \qquad \begin{array}{c}\Delta\ \ [B^y]\\ \vdots\\ C\end{array}}{C}\,(\oplus_{\mathcal{E}})_{x,y}$$

normalizes to

$$\begin{array}{c} \Gamma \\ \vdots \\ B \qquad \Delta \\ \vdots \\ C. \end{array}$$

- *Promotion* followed by *Dereliction*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ !A_1\end{array} \;\dots\; \begin{array}{c}\vdots\\ !A_n\end{array} \qquad \begin{array}{c}[\![!A_1\dots !A_n]\!]\\ \vdots\\ B\end{array}}{!B}\,\mathit{Promotion}}{B}\,\mathit{Dereliction}$$

normalizes to

$$\begin{array}{c} \begin{array}{c}\vdots \\ [\![!A_1\end{array} \;\dots\; \begin{array}{c}\vdots \\ !A_n]\!]\end{array} \\ \vdots \\ B. \end{array}$$

- *Promotion* with *Weakening*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ !A_1\end{array} \;\dots\; \begin{array}{c}\vdots\\ !A_n\end{array} \qquad \begin{array}{c}[\![!A_1\dots !A_n]\!]\\ \vdots\\ B\end{array}}{!B}\,\mathit{Promotion} \qquad \begin{array}{c}\vdots\\ C\end{array}}{C}\,\mathit{Weakening}$$

normalizes to

$$\dfrac{\begin{array}{c}\vdots\\ !A_1\end{array} \;\dots\; \begin{array}{c}\vdots\\ !A_n\end{array} \quad \begin{array}{c}\vdots\\ C\end{array}}{C}\,\mathit{Weakening}^*.$$

- *Promotion* with *Contraction*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ !A_1\end{array} \;\dots\; \begin{array}{c}\vdots\\ !A_n\end{array} \qquad \begin{array}{c}[\![!A_1\dots !A_n]\!]\\ \vdots\\ B\end{array}}{!B}\,\mathit{Prom} \qquad \begin{array}{c}[!B]\;[!B]\\ \vdots\\ C\end{array}}{C}\,\mathit{Cont}$$

normalizes to

$$\dfrac{\begin{array}{c} \dfrac{[!A_1]\dots[!A_n] \qquad \begin{array}{c}[\![!A_1\dots !A_n]\!]\\ \vdots\\ B\end{array}}{!B}\,\mathit{Prom} \quad \dfrac{[!A_1]\dots[!A_n] \qquad \begin{array}{c}[\![!A_1\dots !A_n]\!]\\ \vdots\\ B\end{array}}{!B}\,\mathit{Prom} \\ \vdots \\ C \end{array} \qquad \begin{array}{c}\vdots\\ !A_1\end{array} \;\dots\; \begin{array}{c}\vdots\\ !A_n\end{array}}{C}\,\mathit{Cont}^*.$$

(We have introduced a shorthand notation in the last two cases, where *Weakening** and *Contraction** represent multiple applications of the *Weakening* and *Contraction* rule respectively.) The process above gives rise to a relation which we shall denote by $\rightsquigarrow_\beta$. We shall describe one of the steps as a *β-reduction* rule and say that a deduction is *β-reducible* if we can apply one of the β-reduction rules.

Definition 6. A deduction $\mathcal{D}$ is said to be in *β-normal form* if it is not β-reducible.

2.2 Subformula Property and Commuting Conversions

We have already considered the subformula property in §1.3, in the context of the sequent calculus formulation. Whereas in that case the property held simply by inspection of the derivation rules, things are more delicate in the natural deduction formulation. For example, consider a deduction which ends with an application of the $\multimap_\mathcal{E}$ rule

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ A\multimap B\end{array}\qquad\begin{array}{c}\vdots\,\mathcal{D}_2\\ A\end{array}}{B}\,(\multimap_\mathcal{E}).$$

Even if the deduction is in β-normal form, it is not simply the case by inspection that the subformula property holds, i.e. that $A\multimap B$ is a subformula of Γ or Δ (it is certainly not a subformula of B!). We also note that for most of the elimination rules the conclusion is *not* a subformula of its major premise.[8] These rules are $(\otimes_\mathcal{E})$, $(I_\mathcal{E})$, $(\oplus_\mathcal{E})$, $(\mathbf{f}_\mathcal{E})$, *Weakening* and *Contraction*; which we shall refer to collectively as *bad* eliminations, with those remaining being known as *good* eliminations.

We shall introduce the notion of a *path* through a deduction. The idea is that we trace downwards through a deduction from an assumption, with the hope that each trace yields a path with the property that the every formula is a subformula of either an open assumption or of the final formula in the path. Our notion of a path is based on Prawitz's treatment of **IL** [62], although he attributes the idea to Martin-Löf.

Definition 7. A *path* in a β-normal deduction, $\mathcal{D}$, is a sequence of formulae, $A_0, \ldots, A_n$, such that

1. A_0 is an open assumption or axiom; and
2. A_{i+1} follows A_i if
   (a) A_{i+1} is the conclusion of an introduction rule (excluding $(\mathbf{t}_\mathcal{I})$) and A_i is a premise of the rule (if the rule is *Promotion*, then A_i must not be a minor premise); or
   (b) A_{i+1} is the conclusion of an elimination rule and A_i is either the major premise of a good elimination rule, or the minor premise of a bad elimination rule (excluding $(\mathbf{f}_\mathcal{E})$); or
   (c) A_i is the major premise of a bad elimination rule (excluding $(\mathbf{f}_\mathcal{E})$) and A_{i+1} is an assumption discharged by that application; or
   (d) A_i is the minor premise of an application of the *Promotion* rule and A_{i+1} is the corresponding assumption discharged by that assumption; and
3. A_n is either the conclusion of $\mathcal{D}$, or the major premise of $(I_\mathcal{E})$ or *Weakening*, or a premise of $(\mathbf{f}_\mathcal{E})$.

We shall identify a particular path as mentioned earlier.

Definition 8. A *subformula path* is a path in a deduction, $\mathcal{D}$, such that every formula in it is either a subformula of an assumption or of the final formula in the path.

Consider a β-normal deduction, $\mathcal{D}$, of the form

[8] Girard [34] calls such conclusions *parasitic formulae*.

$$\dfrac{\begin{array}{c}\vdots\\ A\otimes B\end{array} \qquad \begin{array}{c}[A]\ [B]\\ \vdots\\ C\end{array}}{C}\,(\otimes_{\mathcal{E}}).$$

A path through $\mathcal{D}$ is of the form $\ldots, A\otimes B, A, \ldots C, C$ (recall that as $\mathcal{D}$ is in β-normal form then $A\otimes B$ cannot be the result of an introduction rule). As we have mentioned before, the problematic feature of this deduction is the appearance of C in the path. This formula need not have any relation to the formula being eliminated ($A\otimes B$). Importantly it may be that some of the formulae in the path are subformulae of C rather than of any open assumption. If the conclusion of $\mathcal{D}$ is subsequently used as the major premise of another elimination rule, it is easy to see that the resulting path may not be a subformula path. Consider the deduction

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\otimes B\end{array} \qquad \begin{array}{c}[A]\ [B]\\ \vdots\\ C\multimap D\end{array}}{C\multimap D}\,(\otimes_{\mathcal{E}}) \qquad \begin{array}{c}\vdots\\ C\end{array}}{D}\,(\multimap_{\mathcal{E}}),$$

then a path through this deduction is $\ldots, A\otimes B, A, \ldots, C\multimap D, D$. Clearly the formula $C\multimap D$ need not be a subformula of an open assumption (as it could be the conclusion of an introduction rule) and so the path would *not* be a subformula path.

We shall follow the standard solution and introduce additional reductions to remove these problematic occurrences. These occurrences are when the conclusion of a bad elimination rule is the major premise of another elimination rule. We shall use the shorthand notation of Girard [34, Chapter 10] and write

$$\dfrac{C \qquad \vdots}{D}\,(r_{\mathcal{E}}),$$

to denote an elimination rule (r) with the major premise C and conclusion D, where the ellipses represent possible other formulae. This notation covers the ten elimination rules: $(\multimap_{\mathcal{E}})$, $(I_{\mathcal{E}})$, $(\otimes_{\mathcal{E}})$, $(\&_{\mathcal{E}-1})$, $(\&_{\mathcal{E}-2})$, $(\oplus_{\mathcal{E}})$, $(\mathbf{f}_{\mathcal{E}})$, *Dereliction*, *Contraction*, and *Weakening*. We shall follow Girard and commute the r rule upwards, although it should be noted that it would be perfectly admissible (where applicable) to direct these commutations in the other direction.

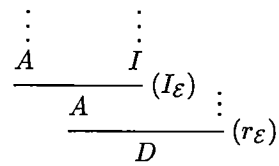
- Commutation of $(\otimes_{\mathcal{E}})$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\otimes B\end{array} \qquad \begin{array}{c}[A]\ \ [B]\\ \vdots\\ C\end{array}}{C}\,(\otimes_{\mathcal{E}}) \qquad \vdots}{D}\,(r_{\mathcal{E}})$$

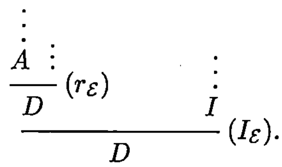
which commutes to

$$\dfrac{\begin{array}{c}\vdots\\ A\otimes B\end{array} \qquad \begin{array}{c}[A]\ \ [B]\\ \vdots\\ \dfrac{C \qquad \vdots}{D}\,(r_{\mathcal{E}})\end{array}}{D}\,(\otimes_{\mathcal{E}}).$$

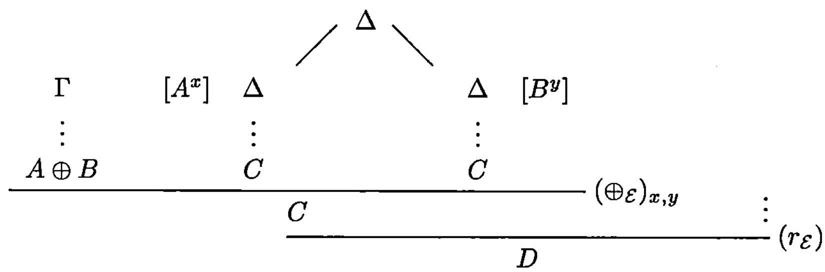
- Commutation of ($I_\mathcal{E}$).



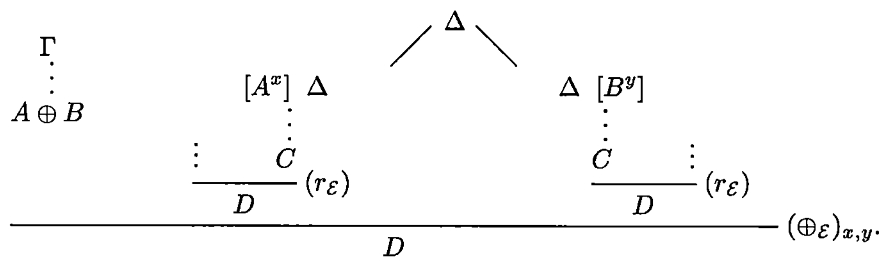
which commutes to



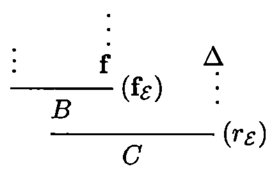
- Commutation of ($\oplus_\mathcal{E}$).



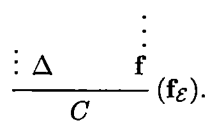
which commutes to



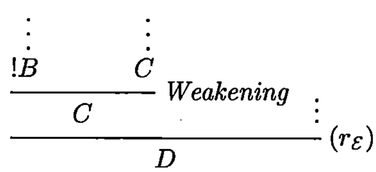
- Commutation of ($\mathbf{f}_\mathcal{E}$).



which commutes to



- Commutation of *Weakening*.



which commutes to

$$\dfrac{\begin{array}{c}\vdots\\ !B\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ C\end{array} \quad \vdots}{D}\,(r_{\mathcal{E}})}{D}\ \textit{Weakening}.$$

- Commutation of *Contraction.*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ !B\end{array} \qquad \begin{array}{c}[!B]\ \ [!B]\\ \vdots\\ C\end{array}}{C}\ \textit{Contraction} \qquad \vdots}{D}\,(r_{\mathcal{E}})$$

which commutes to

$$\dfrac{\begin{array}{c}\vdots\\ !B\end{array} \qquad \dfrac{\begin{array}{c}[!B]\ \ [!B]\\ \vdots\\ C\end{array} \quad \vdots}{D}\,(r_{\mathcal{E}})}{D}\ \textit{Contraction}.$$

However there are still a number of cases to consider before being able to deduce the subformula property. Consider the $(\mathbf{f}_{\mathcal{E}})$ rule

$$\dfrac{\begin{array}{c}\vdots\\ \mathbf{f}\end{array} \qquad \begin{array}{c}\vdots\\ A_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ A_n\end{array}}{B}\,(\mathbf{f}_{\mathcal{E}}).$$

The problem is with the minor premises, $A_1, \ldots, A_n$. We do not necessarily have that these are subformulae of an open assumption. For example, consider the simple (β-normal) deduction

$$\dfrac{\dfrac{C \multimap \mathbf{f} \quad C}{\mathbf{f}}\,(\multimap_{\mathcal{E}}) \qquad \dfrac{A \quad B}{A \otimes B}\,(\otimes_{\mathcal{I}})}{C}\,(\mathbf{f}_{\mathcal{E}}),$$

where clearly $A \otimes B$ is not a subformula of an open assumption nor of C. Thus we impose the (rather strong) reduction, that a deduction of the form

$$\dfrac{\begin{array}{c}\vdots\\ \mathbf{f}\end{array} \qquad \begin{array}{c}\Gamma_1\\ \vdots\\ A_1\end{array}\ \ldots\ \begin{array}{c}\Gamma_n\\ \vdots\\ A_n\end{array}}{B}\,(\mathbf{f}_{\mathcal{E}})$$

commutes to

$$\dfrac{\begin{array}{c}\vdots\\ \mathbf{f}\end{array} \qquad \Gamma_1 \ldots \Gamma_n}{B}\,(\mathbf{f}_{\mathcal{E}}).$$

Similar reasoning for the $(\mathbf{t}_{\mathcal{I}})$ yields the rule that a deduction of the form

$$\frac{\begin{matrix}\Gamma_1 & & \Gamma_n\\ \vdots & & \vdots \\ A_1 & \dots & A_n\end{matrix}}{\mathbf{t}}\,(\mathbf{t}_{\mathcal{I}})$$

commutes to

$$\frac{\Gamma_1\dots\Gamma_n}{\mathbf{t}}\,(\mathbf{t}_{\mathcal{I}}).$$

In fact, we could have been more refined for these two cases. If the minor premises were the conclusion of an elimination rule, then it *would* be the case that it was a subformula of an open assumption. We shall keep these more global rules as we shall see later that they correspond more closely with the steps in the cut elimination process.

Now consider the *Promotion* rule (the introduction rule for the exponential)

$$\frac{\begin{matrix} & & & [\![!A_1 \cdots !A_n]\!] \\ \vdots & & \vdots & \vdots \\ !A_1 & \dots & !A_n & B\end{matrix}}{!B}\ \textit{Promotion}.$$

We have a problem here as we do not know immediately that the $!A_i$ are subformulae of an assumption or of $!B$. Consider a deduction in β-normal form ending with an application of the *Promotion* rule. If $!A_i$ is the conclusion of a bad elimination then a similar argument to that used earlier gives that the path need not be a subformula path.[9] For example, consider the β-normal deduction

$$\dfrac{\dfrac{!C\otimes!(C\multimap A\multimap B) \quad \dfrac{[!C]\quad[!(C\multimap A\multimap B)] \quad \dfrac{\dfrac{[!C]}{C}\,Der \quad \dfrac{[!(C\multimap A\multimap B)]}{C\multimap A\multimap B}\,Der}{A\multimap B}\,(\multimap_{\mathcal{E}})}{!(A\multimap B)}\,Prom}{!(A\multimap B)}\,(\otimes_{\mathcal{E}}) \quad !A \quad \dfrac{\dfrac{[!(A\multimap B)]}{A\multimap B}\,Der \quad \dfrac{[!A]}{A}\,Der}{B}\,(\multimap_{\mathcal{E}})}{!B}\,Prom.$$

Clearly the formula $!(A\multimap B)$ is not a subformula of either an open assumption nor of of the conclusion. As before we shall introduce further commuting conversions to eliminate the problematic occurrence when the conclusion of a bad elimination rule is a minor premise of an application of the *Promotion* rule. Thus a deduction of the form

$$\dfrac{\begin{matrix}\vdots \\ !A_1\end{matrix} \quad \dots \quad \dfrac{\begin{matrix}\vdots \\ C\end{matrix} \quad \begin{matrix}\vdots \\ !A_i\end{matrix}}{!A_i}\,(bad_{\mathcal{E}}) \quad \dots \quad \begin{matrix}\vdots \\ !A_n\end{matrix} \quad \begin{matrix}[\![!A_1\cdots !A_n]\!] \\ \vdots \\ B\end{matrix}}{!B}\ \textit{Promotion}$$

commutes to

$$\dfrac{\begin{matrix}\vdots \\ C\end{matrix} \quad \dfrac{\begin{matrix}\vdots \\ !A_1\end{matrix} \quad \dots \quad \begin{matrix}\vdots \\ !A_i\end{matrix} \quad \dots \quad \begin{matrix}\vdots \\ !A_n\end{matrix} \quad \begin{matrix}[\![!A_1\cdots !A_n]\!] \\ \vdots \\ B\end{matrix}}{!B}\ \textit{Promotion}}{!B}\,(bad_{\mathcal{E}}).$$

[9]The behaviour of the *Promotion* rule with respect to the subformula property was omitted from an earlier paper [16].

We have yet one further possibility; that a minor premise $!A_i$ is the conclusion of another application of the *Promotion* rule (obviously it can't be the result of any other introduction rule). Thus we have another commutation rule, where

$$\dfrac{\begin{array}{c}\vdots\\ !A_1\end{array}\ \cdots\ \dfrac{\begin{array}{c}\vdots\\ !C_1\end{array}\ \cdots\ \begin{array}{c}\vdots\\ !C_m\end{array}\quad \begin{array}{c}[\![!C_1 \ \cdots\ !C_m]\!]\\ \vdots\\ A_i\end{array}}{!A_i}\,Prom\ \cdots\ \begin{array}{c}\vdots\\ !A_n\end{array}\qquad \begin{array}{c}[\![!A_1\ \cdots\ !A_n]\!]\\ \vdots\\ B\end{array}}{!B}\,Prom$$

commutes to

$$\dfrac{\begin{array}{c}\vdots\\ !A_1\end{array}\ \cdots\ \begin{array}{c}\vdots\\ !C_1\end{array}\ \cdots\ \begin{array}{c}\vdots\\ !C_m\end{array}\ \cdots\ \begin{array}{c}\vdots\\ !A_n\end{array}\qquad \begin{array}{c}[\![!A_1\cdots!A_{i-1}]\!]\quad \dfrac{[\![!C_1\cdots!C_m]\!]\quad \begin{array}{c}[\![!C_1\cdots!C_m]\!]\\ \vdots\\ A_i\end{array}}{!A_i}\,Prom \quad [\![!A_{i+1}\cdots!A_n]\!]\\ \vdots\\ B\end{array}}{!B}\,Prom.$$

The commutation process described in this section gives rise to a relation which we denote by $\leadsto_c$. We describe a step as a *commuting conversion*.[10]

Definition 9. A deduction $\mathcal{D}$ is said to be *c-normal form* if no commuting conversions apply.

We can then combine our two notions of a normal form to define a third.

Definition 10. A deduction $\mathcal{D}$ is said to be in (β, c)*-normal form* if it is in both β-normal form and in c-normal form.

Before proving the subformula property, we shall consider the form of a path. It is easy to see that given Definition 7, the bad elimination rules give repeated formula occurrences in a path. A sequence of repeated occurrences of a formula in this way, we shall call a *segment*. Clearly every path can then be uniquely divided into consecutive segments (normally consisting of a single formula occurrence). In other words a path, π, can be expressed as a sequence of segments $\sigma_0, \ldots, \sigma_n$. As a segment represents a sequence of the same formula occurrence, we shall often speak of a segment being a premise or a subformula of another without confusion. We can now show an important property of a path in a (β, c)-normal deduction.

Theorem 4. Let $\mathcal{D}$ be a (β, c)-normal deduction and π be a path in $\mathcal{D}$ and let $\sigma_0, \ldots, \sigma_n$ be a sequence of segments of π. Then there is a segment, σ_i, which separates the path into two (possibly empty) parts, I and E, such that

1. each σ_j in the E part (i.e. $j < i$) is a major premise of an elimination rule and σ_{j+1} is a subformula of σ_j; and
2. σ_i is a premise of an introduction rule or the major premise of *Weakening* or $(I_\mathcal{E})$ or a premise of $(\mathbf{f}_\mathcal{E})$; and
3. each σ_j in the I part, except the last one, (i.e. $i < j < n$) is a premise of an introduction rule and is a subformula of σ_{j+1}.

Proof. By inspection of the deduction $\mathcal{D}$. ∎

Now, essentially as a corollary, we can show that every path is a subformula path, i.e. the subformula property holds.

[10] They are called *permutative conversions* by Prawitz [61].

Theorem 5. Let $\mathcal{D}$ be a deduction in (β, c)-normal form, then every formula in $\mathcal{D}$ is a subformula of the conclusion or an assumption.

Proof. We shall define an order on a path in a deduction. A path which ends with the conclusion of the deduction has the order 1. A path which ends as the major premise of $(I_\mathcal{E})$, *Weakening* or $(\mathbf{f}_\mathcal{E})$, or the minor premise of $(\multimap_\mathcal{E})$ has the order $n+1$ if the minor, or major premise, have the order n, respectively.

We can then show that every path is a subformula path by complete induction on the order of the path, and by use of Theorem 4.

■

3 Axiomatic Formulation

As we have mentioned before an axiomatic formulation of a logic consists of a set of formulae which are known as axioms and a number of derivation rules. Although constructing derivations in such a formulation is somewhat tortuous and unnatural, we can give three important reasons for their study (the first two are used by Hodges [40]).

1. They are very simple to describe. For example, the formulation of **IL** (given in §5) can be described using just one derivation rule (*Modus Ponens*).
2. The logics can be weakened or strengthened by simply removing or adding to the set of axioms. This is certainly the case in the study of Modal Logics (see, for example, Goré's thesis [36]) where it is common to define a logic by first giving its axiomatic formulation.
3. They can be considered as an effective implementation technique for functional programming languages. This technique is due to Turner [77]; Stoye [71] demonstrated how a purpose built machine could be based upon this idea.

The derivation of an axiomatic formulation is reasonably straightforward; a simple guide is given by Hodges [40]. The axioms can often be read off from the natural deduction formulation. For example, consider the introduction and elimination rules for the Tensor, *viz.*

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes_\mathcal{I}), \quad \text{and} \quad \frac{\Gamma \vdash A \otimes B \qquad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}\,(\otimes_\mathcal{E}).$$

The introduction rule suggests an axiom of the form $A \multimap (B \multimap A \otimes B)$, and the elimination rule suggests an axiom $A \otimes B \multimap ((A \multimap (B \multimap C)) \multimap C)$. However it should be noted that this technique only works for so-called *pure* logical rules; namely those which contain no side-conditions. Thus for the additives and the exponential this simple technique does not apply, and we have to call upon experience with other logics and consider certain properties which enable us to derive an appropriate set of axioms.

It should be noted that both Avron [7] and Troelstra [75, Chapter 7] have considered an axiomatic formulation of **ILL**. The axiomatic formulation is given in Figure 2.4.[11] In what follows we shall sometimes write an instance of an axiom as $\vdash \mathsf{c}$.

The *Promotion* rule has the side condition that the context must be *empty* when it is applied. This is tradition for modal logics and for **ILL** it seems to be unavoidable. It is worth noting the difficulty with representing the additive disjunction, $\oplus$. There seems to be no way of defining the connective other than with reference to the With connective (in the axiom `sdist`). This seems to be a weakness with this particular system rather than the logic itself. It is not clear if this problem is eliminated when considering Hilbert-style presentations of *Classical* Linear Logic (such as that of Hesselink [38]).

[11] As mentioned in §1 some of the exponential rules are similar to those from Modal Logic. In the axiomatic formulation we see the similarities as well. The axiom `eps` is similar to the modal axiom $\mathbf{T}: \Box A \supset A$, `delta` is similar to $\mathbf{4}: \Box A \supset \Box\Box A$ and `edist` is similar to $\mathbf{K}: \Box(A \supset B) \supset (\Box A \supset \Box B)$.

$$
\begin{array}{rcl}
\texttt{I} & : & A \multimap A \\
\texttt{B} & : & (B \multimap C) \multimap ((A \multimap B) \multimap (A \multimap C)) \\
\texttt{C} & : & (A \multimap (B \multimap C)) \multimap (B \multimap (A \multimap C)) \\
\\
\texttt{tensor} & : & A \multimap (B \multimap A \otimes B) \\
\texttt{split} & : & A \otimes B \multimap ((A \multimap (B \multimap C)) \multimap C) \\
\\
\texttt{unit} & : & I \\
\texttt{let} & : & A \multimap (I \multimap A) \\
\\
\texttt{fst} & : & A \& B \multimap A \\
\texttt{snd} & : & A \& B \multimap B \\
\texttt{wdist} & : & (A \multimap B) \& (A \multimap C) \multimap (A \multimap B \& C) \\
\\
\texttt{inl} & : & A \multimap A \oplus B \\
\texttt{inr} & : & B \multimap A \oplus B \\
\texttt{sdist} & : & (A \multimap C) \& (B \multimap C) \multimap (A \oplus B \multimap C) \\
\\
\texttt{init} & : & \mathbf{f} \multimap A \\
\texttt{term} & : & A \multimap \mathbf{t} \\
\\
\texttt{dupl} & : & (!A \multimap (!A \multimap B)) \multimap (!A \multimap B) \\
\texttt{disc} & : & B \multimap (!A \multimap B) \\
\texttt{eps} & : & !A \multimap A \\
\texttt{delta} & : & !A \multimap !!A \\
\texttt{edist} & : & !(A \multimap B) \multimap (!A \multimap !B)
\end{array}
$$

$$
\frac{}{A \vdash A}\ \textit{Identity} \qquad \frac{}{\vdash A}\ \textit{Axiom} \quad \text{(where } \texttt{c} : A \text{ is taken from above)}
$$

$$
\frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ \textit{Modus Ponens} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \textit{With}
$$

$$
\frac{\vdash A}{\vdash !A}\ \textit{Promotion}
$$

Figure 2.4: Axiomatic Formulation of **ILL**

An important theorem is the so-called *Deduction Theorem*, which in essence allows us to 'remove' an assumption from a derivation. In other words, given any derivation, we can remove its dependence upon any assumptions. We find that such a theorem holds for our formulation.

Theorem 6. If $\Gamma, A \vdash B$ then $\Gamma \vdash A \multimap B$.

Proof. By induction on the structure of the derivation $\Gamma, A \vdash B$.

- A derivation of the form

$$\frac{}{A \vdash A}\ \textit{Identity}$$

  is transformed to

$$\frac{}{\vdash A \multimap A}\ \textit{Axiom.}$$

- A derivation of the form

$$\frac{\Gamma, A \vdash B \multimap C \qquad \Delta \vdash B}{\Gamma, \Delta, A \vdash C}\ \textit{Modus Ponens}$$

  is transformed to

$$\dfrac{\dfrac{\dfrac{}{\vdash (A \multimap (B \multimap C)) \multimap (B \multimap (A \multimap C))} \quad \Gamma \vdash A \multimap (B \multimap C)}{\Gamma \vdash B \multimap (A \multimap C)}\ \textit{M.P.} \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \multimap C}\ \textit{M.P.}$$

- A derivation of the form

$$\frac{\Gamma \vdash B \multimap C \qquad \Delta, A \vdash B}{\Gamma, \Delta, A \vdash C}\ \textit{Modus Ponens}$$

  is transformed to

$$\dfrac{\dfrac{\dfrac{}{\vdash (B \multimap C) \multimap ((A \multimap B) \multimap (A \multimap C))} \quad \Gamma \vdash B \multimap C}{\Gamma \vdash (A \multimap B) \multimap (A \multimap C)}\ \textit{M.P.} \qquad \Delta \vdash A \multimap B}{\Gamma, \Delta \vdash A \multimap B}\ \textit{M.P.}$$

- A derivation of the form

$$\frac{\Gamma, A \vdash B \qquad \Gamma, A \vdash C}{\Gamma, A \vdash B \& C}\ \textit{With}$$

  is transformed to

$$\dfrac{\dfrac{}{\vdash (A \multimap B) \& (A \multimap C) \multimap (A \multimap B \& C)} \qquad \dfrac{\Gamma \vdash A \multimap B \quad \Gamma \vdash A \multimap C}{\Gamma \vdash (A \multimap B) \& (A \multimap C)}\ \textit{With}}{\Gamma \vdash A \multimap B \& C}\ \textit{M.P.}$$

■

In §4.3 we shall make use of the deduction theorem. For conciseness we shall often write it as if it were a proof rule, *viz.*

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ D.T.$$

4 Comparisons

We would expect there to be a close relationship between the three formulations of **ILL** which we have presented. Indeed we can define procedures to map proofs from one formulation to another. We shall consider each procedure in turn. (For conciseness we shall use the sequent-style presentation of the natural deduction formulation.)

4.1 From Sequent Calculus to Natural Deduction

We shall define a procedure $\mathcal{N}$ by induction on the sequent proof tree, which we shall denote by π.

- The proof

$$\frac{}{A \vdash A}\ \textit{Identity}$$

is mapped to the deduction

$$\frac{}{A \vdash A}\ \textit{Identity}.$$

- A proof π of the form

$$\frac{\begin{array}{c}\pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \Delta, A \vdash B\end{array}}{\Gamma, \Delta \vdash B}\ \textit{Cut}$$

is mapped to the deduction

$$\frac{\begin{array}{c}\mathcal{N}(\pi_1)\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{N}(\pi_2)\\ \Delta, A \vdash B\end{array}}{\Gamma, \Delta \vdash B}\ \textit{Subs.}$$

- A proof π of the form

$$\frac{\begin{array}{c}\pi_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \Delta, B \vdash C\end{array}}{\Gamma, A \multimap B, \Delta \vdash C}\ (\multimap_{\mathcal{L}})$$

is mapped to the deduction

$$\frac{\dfrac{\dfrac{}{A \multimap B \vdash A \multimap B} \qquad \begin{array}{c}\mathcal{N}(\pi_1)\\ \Gamma \vdash A\end{array}}{A \multimap B, \Gamma \vdash B}\ (\multimap_{\mathcal{E}}) \qquad \begin{array}{c}\mathcal{N}(\pi_2)\\ \Delta, B \vdash C\end{array}}{\Gamma, A \multimap B, \Delta \vdash C}\ \textit{Subs.}$$

- A proof π of the form

$$\frac{\begin{array}{c}\pi_1\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{R}})$$

is mapped to the deduction

$$\frac{\begin{array}{c}\mathcal{N}(\pi_1)\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{I}}).$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash A\end{matrix}}{\Gamma, I \vdash A}\,(I_{\mathcal{L}})$$

is mapped to the deduction

$$\dfrac{\begin{matrix}\mathcal{N}(\pi_1)\\ \Gamma \vdash A\end{matrix} \qquad \overline{I \vdash I}}{\Gamma, I \vdash A}\,(I_{\mathcal{E}}).$$

- A proof of the form

$$\dfrac{}{\vdash I}\,(I_{\mathcal{R}})$$

is mapped to the deduction

$$\dfrac{}{\vdash I}\,(I_{\mathcal{I}}).$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Delta, A, B \vdash C\end{matrix}}{\Delta, A\otimes B \vdash C}\,(\otimes_{\mathcal{L}})$$

is mapped to the deduction

$$\dfrac{\overline{A\otimes B \vdash A\otimes B} \qquad \begin{matrix}\mathcal{N}(\pi_1)\\ \Delta, A, B \vdash C\end{matrix}}{A\otimes B, \Delta \vdash C}\,(\otimes_{\mathcal{E}}).$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_2\\ \Delta \vdash B\end{matrix}}{\Gamma, \Delta \vdash A\otimes B}\,(\otimes_{\mathcal{R}})$$

is mapped to the deduction

$$\dfrac{\begin{matrix}\mathcal{N}(\pi_1)\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{N}(\pi_2)\\ \Delta \vdash B\end{matrix}}{\Gamma, \Delta \vdash A\otimes B}\,(\otimes_{\mathcal{I}}).$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma, A \vdash C\end{matrix}}{\Gamma, A\&B \vdash C}\,(\&_{\mathcal{L}-1})$$

is mapped to the deduction

$$\dfrac{\dfrac{\dfrac{}{A\&B \vdash A\&B}\,\mathit{Identity}}{A\&B \vdash A}\,(\&_{\mathcal{E}-1}) \qquad \begin{matrix}\mathcal{N}(\pi_1)\\ \Gamma, A \vdash C\end{matrix}}{\Gamma, A\&B \vdash C}\,\mathit{Subs.}$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma, B \vdash C\end{matrix}}{\Gamma, A\&B \vdash C}\,(\&_{\mathcal{L}-2})$$

is mapped to the deduction

$$\dfrac{\dfrac{\dfrac{}{A\&B \vdash A\&B}\,\textit{Identity}}{A\&B \vdash B}\,(\&_{\mathcal{E}-2}) \qquad \begin{matrix}\mathcal{N}(\pi_1)\\ \Gamma, B \vdash C\end{matrix}}{\Gamma, A\&B \vdash C}\,\textit{Subs.}$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\pi_2\\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A\&B}\,(\&_{\mathcal{R}})$$

is mapped to the deduction

$$\dfrac{\begin{matrix}\mathcal{N}(\pi_1)\\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{N}(\pi_2)\\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A\&B}\,(\&_{\mathcal{R}}).$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma, A \vdash C\end{matrix} \qquad \begin{matrix}\pi_2\\ \Gamma, B \vdash C\end{matrix}}{\Gamma, A \oplus B \vdash C}\,(\oplus_{\mathcal{L}})$$

is mapped to the deduction

$$\dfrac{\dfrac{}{A \oplus B \vdash A \oplus B}\,\textit{Identity} \qquad \begin{matrix}\mathcal{N}(\pi_1)\\ \Gamma, A \vdash C\end{matrix} \qquad \begin{matrix}\mathcal{N}(\pi_2)\\ \Gamma, B \vdash C\end{matrix}}{\Gamma, A \oplus B \vdash C}\,(\oplus_{\mathcal{E}}).$$

- A proof π of the form

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{R}-1})$$

is mapped to the deduction

$$\dfrac{\begin{matrix}\mathcal{N}(\pi_1)\\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{I}-1}).$$

- A proof π of the form

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{R}-2})$$

is mapped to the deduction

$$\dfrac{\begin{array}{c}\mathcal{N}(\pi_1)\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{I}-2}).$$

- A proof π of the form

$$\dfrac{}{\Gamma \vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{R}})$$

is mapped to the deduction

$$\dfrac{\dfrac{}{\Gamma \vdash \Gamma}\,Id}{\Gamma \vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{I}}).$$

- A proof π of the form

$$\dfrac{}{\Gamma, \mathbf{f} \vdash A}\,(\mathbf{f}_{\mathcal{L}})$$

is mapped to the deduction

$$\dfrac{\dfrac{}{\Gamma \vdash \Gamma}\,Id \qquad \dfrac{}{\mathbf{f} \vdash \mathbf{f}}\,Id}{\Gamma, \mathbf{f} \vdash A}\,(\mathbf{f}_{\mathcal{E}}).$$

- A proof π of the form

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma \vdash B\end{array}}{\Gamma, !A \vdash B}\,\textit{Weakening}$$

is mapped to the deduction

$$\dfrac{\dfrac{}{!A \vdash !A} \qquad \begin{array}{c}\mathcal{N}(\pi_1)\\ \Gamma \vdash B\end{array}}{\Gamma, !A \vdash B}\,\textit{Weakening}.$$

- A proof π of the form

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma, !A, !A \vdash B\end{array}}{\Gamma, !A \vdash B}\,\textit{Contraction}$$

is mapped to the deduction

$$\dfrac{\dfrac{}{!A \vdash !A} \qquad \begin{array}{c}\mathcal{N}(\pi_1)\\ \Gamma, !A, !A \vdash B\end{array}}{\Gamma, !A \vdash B}\,\textit{Contraction}.$$

- A proof π of the form

$$\cfrac{\begin{array}{c}\pi_1\\ \Gamma, A \vdash B\end{array}}{\Gamma, !A \vdash B}\ \textit{Dereliction}$$

is mapped to the deduction

$$\cfrac{\cfrac{\cfrac{}{!A \vdash !A}}{!A \vdash A}\ \textit{Dereliction} \qquad \begin{array}{c}\mathcal{N}(\pi_1)\\ \Gamma, A \vdash B\end{array}}{\Gamma, !A \vdash B}\ \textit{Subs.}$$

- Finally, a proof π of the form

$$\cfrac{\begin{array}{c}\pi_1\\ !A_1, \ldots, !A_n \vdash B\end{array}}{!A_1, \ldots, !A_n \vdash !B}\ \textit{Promotion}$$

is mapped to the deduction

$$\cfrac{\cfrac{}{!A_1 \vdash !A_1} \quad \cdots \quad \cfrac{}{!A_n \vdash !A_n} \qquad \begin{array}{c}\mathcal{N}(\pi_1)\\ !A_1, \ldots, !A_n \vdash B\end{array}}{!A_1, \ldots, !A_n \vdash !B}\ \textit{Promotion.}$$

4.2 From Natural Deduction to Sequent Calculus

We shall define a procedure $\mathcal{S}$ by induction on the deduction tree, which we shall denote by $\mathcal{D}$.

- The deduction

$$\cfrac{}{A \vdash A}\ \textit{Identity}$$

is mapped to the proof

$$\cfrac{}{A \vdash A}\ \textit{Identity.}$$

- The deduction $\mathcal{D}$ of the form

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{I}})$$

is mapped to the proof

$$\cfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{R}}).$$

- A deduction $\mathcal{D}$ of the form

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A \multimap B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta \vdash A\end{array}}{\Gamma, \Delta \vdash B}\ (\multimap_{\mathcal{E}})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma \vdash A \multimap B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_2)\\ \Delta \vdash A\end{array} \qquad \overline{B \vdash B}}{A \multimap B, \Delta \vdash B}\,(\multimap_{\mathcal{L}})}{\Gamma, \Delta \vdash B}\,Cut.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{}{\vdash I}\,(I_{\mathcal{I}})$$

is mapped to the sequent

$$\dfrac{}{\vdash I}\,(I_{\mathcal{R}}).$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta \vdash I\end{array}}{\Gamma, \Delta \vdash A}\,(I_{\mathcal{E}})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_2)\\ \Delta \vdash I\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma \vdash A\end{array}}{\Gamma, I \vdash A}\,(I_{\mathcal{L}})}{\Gamma, \Delta \vdash A}\,Cut.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes_{\mathcal{I}})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{S}(\mathcal{D}_2)\\ \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes_{\mathcal{R}}).$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A \otimes B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta, A, B \vdash C\end{array}}{\Gamma, \Delta \vdash C}\,(\otimes_{\mathcal{E}})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma \vdash A \otimes B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_2)\\ \Delta, A, B \vdash C\end{array}}{\Delta, A \otimes B \vdash C}\,(\otimes_{\mathcal{L}})}{\Gamma, \Delta \vdash C}\,Cut.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma \vdash A & \Gamma \vdash B\end{array}}{\Gamma \vdash A\&B}\ (\&_{\mathcal{I}})$$

is mapped to the proof

$$\dfrac{\begin{array}{cc}\mathcal{S}(\mathcal{D}_1) & \mathcal{S}(\mathcal{D}_2) \\ \Gamma \vdash A & \Gamma \vdash B\end{array}}{\Gamma \vdash A\&B}\ (\&_{\mathcal{R}}).$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \vdash A\&B\end{array}}{\Gamma \vdash A}\ (\&_{\mathcal{E}-1})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1) \\ \Gamma \vdash A\&B\end{array} \qquad \dfrac{\dfrac{}{A \vdash A}\ \mathit{Identity}}{A\&B \vdash A}\ (\&_{\mathcal{L}-1})}{\Gamma \vdash A}\ \mathit{Cut.}$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \vdash A\&B\end{array}}{\Gamma \vdash B}\ (\&_{\mathcal{E}-2})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1) \\ \Gamma \vdash A\&B\end{array} \qquad \dfrac{\dfrac{}{B \vdash B}\ \mathit{Identity}}{A\&B \vdash B}\ (\&_{\mathcal{L}-2})}{\Gamma \vdash B}\ \mathit{Cut.}$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{I}-1})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1) \\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{R}-1}).$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{I}-2})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1) \\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{R}-2}).$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\oplus B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta, A \vdash C\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Delta, B\vdash C\end{array}}{\Gamma,\Delta\vdash C}\,(\oplus_{\mathcal{E}})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma\vdash A\oplus B\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_2)\\ \Delta,A\vdash C\end{array}\qquad \begin{array}{c}\mathcal{S}(\mathcal{D}_3)\\ \Delta,B\vdash C\end{array}}{\Delta, A\oplus B\vdash C}\,(\oplus_{\mathcal{L}})}{\Gamma,\Delta\vdash C}\,\mathit{Cut}.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1\vdash A_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n\\ \Gamma_n\vdash A_n\end{array}}{\Gamma_1,\ldots,\Gamma_n\vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{I}})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma_1\vdash A_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{S}(\mathcal{D}_n)\\ \Gamma_n\vdash A_n\end{array} \quad \dfrac{}{A_1,\ldots,A_n\vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{R}}).}{\Gamma_1,\ldots,\Gamma_n\vdash \mathbf{t}}\,\mathit{Cut}_n$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1\vdash A_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n\\ \Gamma_n\vdash A_n\end{array} \qquad \begin{array}{c}\mathcal{D}_{n+1}\\ \Delta\vdash \mathbf{f}\end{array}}{\Gamma_1,\ldots,\Gamma_n,\Delta\vdash A}\,(\mathbf{f}_{\mathcal{E}})$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma_1\vdash A_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{S}(\mathcal{D}_n)\\ \Gamma_n\vdash A_n\end{array} \qquad \begin{array}{c}\mathcal{S}(\mathcal{D}_{n+1})\\ \Delta\vdash \mathbf{f}\end{array} \qquad \dfrac{}{A_1,\ldots,A_n,\mathbf{f}\vdash A}\,(\mathbf{f}_{\mathcal{L}})}{\Gamma,\Delta\vdash A}\,\mathit{Cut}_{n+2}.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash\, !A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta\vdash B\end{array}}{\Gamma,\Delta\vdash B}\,\mathit{Weakening}$$

is mapped to the proof

$$\dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_1)\\ \Gamma\vdash\, !A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{S}(\mathcal{D}_2)\\ \Delta\vdash B\end{array}}{\Delta, !A\vdash B}\,\mathit{Weakening}}{\Gamma,\Delta\vdash B}\,\mathit{Cut}.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma \vdash\, !A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Delta, !A, !A \vdash B\end{matrix}}{\Gamma, \Delta \vdash B}\;\textit{Contraction}$$

  is mapped to the proof

$$\dfrac{\begin{matrix}\mathcal{S}(\mathcal{D}_1)\\ \Gamma \vdash\, !A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{S}(\mathcal{D}_2)\\ \Delta, !A, !A \vdash B\end{matrix}}{\Delta, !A \vdash B}\;\textit{Contraction}}{\Gamma, \Delta \vdash B}\;\textit{Cut}.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma \vdash\, !A\end{matrix}}{\Gamma \vdash A}\;\textit{Dereliction}$$

  is mapped to the proof

$$\dfrac{\begin{matrix}\mathcal{S}(\mathcal{D}_1)\\ \Gamma \vdash\, !A\end{matrix} \qquad \dfrac{\overline{A \vdash A}}{!A \vdash A}\;\textit{Dereliction}}{\Gamma \vdash A}\;\textit{Cut}.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Delta_1 \vdash\, !A_1\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_n\\ \Delta_n \vdash\, !A_n\end{matrix} \qquad \begin{matrix}\mathcal{D}_{n+1}\\ !A_1, \ldots, !A_n \vdash B\end{matrix}}{\Delta_1, \ldots, \Delta_n \vdash\, !B}\;\textit{Promotion}$$

  is mapped to the proof

$$\dfrac{\begin{matrix}\mathcal{S}(\mathcal{D}_1)\\ \Delta_1 \vdash\, !A_1\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{S}(\mathcal{D}_n)\\ \Delta_n \vdash\, !A_n\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{S}(\mathcal{D}_{n+1})\\ !A_1, \ldots, !A_n \vdash B\end{matrix}}{!A_1, \ldots, !A_n \vdash\, !B}\;\textit{Promotion}}{\Delta_1, \ldots, \Delta_n \vdash\, !B}\;\textit{Cut}_n.$$

4.3 From Natural Deduction to Axiomatic

We shall define a procedure $\mathcal{H}$ by induction on the deduction tree, which we shall denote by $\mathcal{D}$.

- The deduction

$$\dfrac{}{A \vdash A}\;\textit{Identity}$$

  is mapped to the derivation

$$\dfrac{}{A \vdash A}\;\textit{Identity}$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \multimap B}\,(\multimap_{\mathcal{I}})$$

is mapped to the derivation

$$\dfrac{\begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \multimap B}\,D.T.$$

- A deduction $\mathcal{D}$ of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A \multimap B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta \vdash A\end{array}}{\Gamma, \Delta \vdash B}\,(\multimap_{\mathcal{E}})$$

is mapped to the derivation

$$\dfrac{\begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash A \multimap B\end{array} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Delta \vdash A\end{array}}{\Gamma, \Delta \vdash B}\,\textit{Modus Ponens.}$$

- The deduction

$$\dfrac{}{\vdash I}\,(I_{\mathcal{I}})$$

is mapped to the derivation

$$\dfrac{}{\vdash I}\,\textit{Axiom.}$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta \vdash I\end{array}}{\Gamma, \Delta \vdash A}\,(I_{\mathcal{E}})$$

is mapped to the derivation

$$\dfrac{\dfrac{\dfrac{}{\vdash A \multimap (I \multimap A)} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash A\end{array}}{\Gamma \vdash I \multimap A}\,M.P. \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Delta \vdash I\end{array}}{\Gamma, \Delta \vdash A}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes_{\mathcal{I}})$$

is mapped to the derivation

$$\dfrac{\dfrac{\dfrac{}{\vdash A\multimap(B\multimap A\otimes B)} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma\vdash A\end{array}}{\Gamma\vdash B\multimap A\otimes B}\,M.P. \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Delta\vdash B\end{array}}{\Gamma,\Delta\vdash A\otimes B}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash A\otimes B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta, A, B\vdash C\end{array}}{\Gamma,\Delta\vdash C}\,(\otimes_{\mathcal{E}})$$

is mapped to the derivation

$$\dfrac{\dfrac{\dfrac{}{\vdash A\otimes B\multimap((A\multimap B\multimap C)\multimap C)} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma\vdash A\otimes B\end{array}}{\Gamma\vdash (A\multimap(B\multimap C))\multimap C} \qquad \dfrac{\begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Delta, A, B\vdash C\end{array}}{\Delta\vdash A\multimap(B\multimap C)}\,D.T.^2}{\Gamma,\Delta\vdash C}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\vdash B\end{array}}{\Gamma\vdash A\& B}\,(\&_{\mathcal{I}})$$

is mapped to the derivation

$$\dfrac{\begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma\vdash A\end{array} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Gamma\vdash B\end{array}}{\Gamma\vdash A\& B}\,With.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash A\& B\end{array}}{\Gamma\vdash A}\,(\&_{\mathcal{E}-1})$$

is mapped to the derivation

$$\dfrac{\dfrac{}{\vdash A\& B\multimap A}\,Axiom \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma\vdash A\& B\end{array}}{\Gamma\vdash A}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash A\& B\end{array}}{\Gamma\vdash B}\,(\&_{\mathcal{E}-2})$$

is mapped to the derivation

$$\dfrac{\dfrac{}{\vdash A\& B\multimap B}\,Axiom \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma\vdash A\& B\end{array}}{\Gamma\vdash B}\,M.P.$$

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{I}-1})$$

is mapped to the derivation

$$\frac{\overline{\vdash A \multimap A \oplus B} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \oplus B}\,M.P.$$

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \oplus B}\,(\oplus_{\mathcal{I}-2})$$

is mapped to the derivation

$$\frac{\overline{\vdash B \multimap A \oplus B} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \oplus B}\,M.P.$$

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A \oplus B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta, A \vdash C\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Delta, B \vdash C\end{array}}{\Gamma, \Delta \vdash C}\,(\oplus_{\mathcal{E}})$$

is mapped to the deduction

$$\frac{\dfrac{\overline{\vdash (A \multimap C)\&(B \multimap C) \multimap (A \oplus B \multimap C)} \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Delta, A \vdash C\end{array}}{\Delta \vdash A \multimap C}\,D.T. \qquad \dfrac{\begin{array}{c}\mathcal{H}(\mathcal{D}_3)\\ \Delta, B \vdash C\end{array}}{\Delta \vdash B \multimap C}\,D.T.}{\Delta \vdash (A \multimap C)\&(B \multimap C)}\,With}{\Delta \vdash A \oplus B \multimap C} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash A \oplus B\end{array}}{\Gamma, \Delta \vdash C}\,M.P.$$

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \vdash A_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n\\ \Gamma_n \vdash A_n\end{array}}{\Gamma_1, \ldots, \Gamma_n \vdash \mathbf{t}}\,(\mathbf{t}_{\mathcal{I}})$$

is mapped to the derivation

$$\frac{\overline{\vdash A_1 \otimes \ldots \otimes A_n \multimap \mathbf{t}} \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\overline{\vdash A \multimap B \multimap A \otimes B} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma_1 \vdash A_1\end{array}}{\Gamma_1 \vdash A_2 \multimap A_1 \otimes A_2}\,M.P. \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Gamma_2 \vdash A_2\end{array}}{}\,M.P.\\ \vdots\end{array}}{\vec{\Gamma} \vdash A_1 \otimes \ldots \otimes A_n}\,M.P.}{\vec{\Gamma} \vdash \mathbf{t}}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \vdash A_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n\\ \Gamma_n \vdash A_n\end{array} \qquad \begin{array}{c}\mathcal{D}_{n+1}\\ \Delta \vdash \mathbf{f}\end{array}}{\Gamma_1,\ldots,\Gamma_n,\Delta \vdash B}\,(\mathbf{f}_{\mathcal{E}})$$

is mapped to the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{f}\multimap(A_1\multimap(\ldots(A_n\multimap B)\ldots))} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_{n+1})\\ \Delta \vdash \mathbf{f}\end{array}}{\Delta \vdash A_1\multimap(\ldots(A_n\multimap B)\ldots)}\,M.P. \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma_1 \vdash A_1\end{array}}{\vdots}\,M.P. \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_n)\\ \Gamma_n \vdash A_n\end{array}}{\vec{\Gamma},\Delta \vdash B}$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash !A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta \vdash B\end{array}}{\Gamma,\Delta \vdash B}\,\textit{Weakening}$$

is mapped to the derivation

$$\dfrac{\dfrac{\dfrac{}{\vdash B\multimap(!A\multimap B)} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Delta \vdash B\end{array}}{\Delta \vdash !A\multimap B} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash !A\end{array}}{\Gamma,\Delta \vdash B}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash !A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta,!A,!A \vdash B\end{array}}{\Gamma,\Delta \vdash B}\,\textit{Contraction}$$

is mapped to the derivation

$$\dfrac{\dfrac{\dfrac{}{\vdash (!A\multimap !A\multimap B)\multimap(!A\multimap B)} \qquad \dfrac{\begin{array}{c}\mathcal{H}(\mathcal{D}_2)\\ \Delta,!A,!A \vdash B\end{array}}{\Delta \vdash !A\multimap !A\multimap B}\,D.T.^2}{\Delta \vdash !A\multimap B}\,M.P. \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash !A\end{array}}{\Gamma,\Delta \vdash B}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash !A\end{array}}{\Gamma \vdash A}\,\textit{Dereliction}$$

is mapped to the derivation

$$\dfrac{\dfrac{}{\vdash !A\multimap A} \qquad \begin{array}{c}\mathcal{H}(\mathcal{D}_1)\\ \Gamma \vdash !A\end{array}}{\Gamma \vdash A}\,M.P.$$

- A deduction of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma_1 \vdash\, !A_1\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_n\\ \Gamma_n \vdash\, !A_n\end{matrix} \qquad \begin{matrix}\mathcal{D}_{n+1}\\ !A_1,\ldots,!A_n \vdash B\end{matrix}}{\Gamma_1,\ldots,\Gamma_n \vdash\, !B}\ \textit{Promotion}$$

is mapped first to the derivation

$$\dfrac{\dfrac{\begin{matrix}\mathcal{H}(\mathcal{D})_{n+1}\\ !A_1,\ldots,!A_n\vdash B\end{matrix}}{\vdash\, !A_1\multimap(!A_2\multimap\ldots\multimap B)}\ D.T.^n}{\vdash\, !(!A_1\multimap(!A_2\multimap\ldots\multimap B))}\ \textit{Prom.}$$

Writing $\mathcal{D}$ for this derivation we can form the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{B}} \quad \dfrac{\dfrac{}{\vdash \mathtt{edist}} \quad \begin{matrix}\mathcal{D}\\ \vdash\, !(!A_1\multimap(!A_2\multimap\ldots\multimap B))\end{matrix}}{\vdash\, !!A_1\multimap!(!A_2\multimap\ldots\multimap B)}}{\vdash (!A_1\multimap!!A_1)\multimap(!A_1\multimap!(!A_2\multimap\ldots\multimap B))}\ \textit{M.P.} \quad \dfrac{}{\vdash\, !A\multimap!!A}}{\vdash\, !A_1\multimap!(!A_2\multimap\ldots\multimap B)}\ \textit{M.P.} \quad \begin{matrix}\mathcal{H}(\mathcal{D}_1)\\ \Gamma_1\vdash\, !A_1\end{matrix}}{\Gamma_1\vdash\, !(!A_2\multimap\ldots\multimap B)}\ \textit{M.P.}$$

$$\vdots$$

$$\dfrac{}{\Gamma_1,\ldots,\Gamma_n\vdash\, !B}\ \textit{M.P.}$$

4.4 From Axiomatic to Natural Deduction

We shall define a procedure $\mathcal{J}$ by induction on the derivation which we shall denote as $\mathcal{D}$. (We shall not consider all the axioms, but as examples, consider those which model the exponential.)

- A derivation of the form

$$\dfrac{}{A\vdash A}\ \textit{Identity}$$

is mapped to the deduction

$$\dfrac{}{A\vdash A}\ \textit{Identity}$$

- A derivation of the form

$$\dfrac{}{\vdash ((!A\multimap!A\multimap B)\multimap(!A\multimap B))}\ \textit{Axiom}$$

is mapped to the following deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{}{!A\vdash\, !A} \quad \dfrac{\dfrac{\dfrac{}{!A\multimap!A\multimap B\vdash\, !A\multimap!A\multimap B} \quad \dfrac{}{!A\vdash\, !A}}{!A\multimap!A\multimap B, !A\vdash\, !A\multimap B}\ (\multimap_{\mathcal{E}}) \quad \dfrac{}{!A\vdash\, !A}}{!A\multimap!A\multimap B, !A, !A\vdash B}\ (\multimap_{\mathcal{E}})}{!A\multimap!A\multimap B, !A\vdash B}\ \textit{Contraction}}{!A\multimap!A\multimap B\vdash\, !A\multimap B}\ (\multimap_{\mathcal{I}})}{\vdash ((!A\multimap!A\multimap B)\multimap(!A\multimap B))}\ (\multimap_{\mathcal{I}}).$$

- A derivation of the form

$$\dfrac{}{\vdash B \multimap !A \multimap B}\ \textit{Axiom}$$

is mapped to the deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{}{!A \vdash !A} \qquad \dfrac{}{B \vdash B}}{B, !A \vdash B}\ \textit{Weakening}}{B \vdash !A \multimap B}\ (\multimap_{\mathcal{I}})}{\vdash B \multimap !A \multimap B}\ (\multimap_{\mathcal{I}}).$$

- A derivation of the form

$$\dfrac{}{\vdash !A \multimap A}\ \textit{Axiom}$$

is mapped to the deduction

$$\dfrac{\dfrac{}{!A \vdash !A}}{!A \vdash A}\ \textit{Dereliction.}$$

- A derivation of the form

$$\dfrac{}{\vdash !A \multimap !!A}\ \textit{Axiom}$$

is mapped to the following deduction

$$\dfrac{\dfrac{\dfrac{}{!A \vdash !A} \qquad \dfrac{}{!A \vdash !A}}{!A \vdash !!A}\ \textit{Promotion}}{\vdash !A \multimap !!A}\ (\multimap_{\mathcal{I}}).$$

- A derivation of the form

$$\dfrac{}{\vdash !(A \multimap B) \multimap (!A \multimap !B)}\ \textit{Axiom}$$

is mapped to the deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{}{!A \vdash !A} \qquad \dfrac{}{!(A \multimap B) \vdash !(A \multimap B)} \qquad \dfrac{\dfrac{\dfrac{}{!(A \multimap B) \vdash !(A \multimap B)}}{!(A \multimap B) \vdash A \multimap B}\ \textit{Der.} \qquad \dfrac{\dfrac{}{!A \vdash !A}}{!A \vdash A}\ \textit{Der.}}{!(A \multimap B), !A \vdash B}\ (\multimap_{\mathcal{E}})}{!(A \multimap B), !A \vdash !B}\ \textit{Promotion}}{!(A \multimap B) \vdash !A \multimap !B}\ (\multimap_{\mathcal{I}})}{\vdash !(A \multimap B) \multimap (!A \multimap !B)}\ (\multimap_{\mathcal{I}}).$$

- A derivation of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \vdash A \multimap B\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \Delta \vdash A\end{matrix}}{\Gamma, \Delta \vdash B}\ \textit{Modus Ponens}$$

is mapped to the deduction

$$\dfrac{\begin{matrix}\mathcal{J}(\mathcal{D}_1) \\ \Gamma \vdash A \multimap B\end{matrix} \qquad \begin{matrix}\mathcal{J}(\mathcal{D}_2) \\ \Delta \vdash A\end{matrix}}{\Gamma, \Delta \vdash B}\ (\multimap_{\mathcal{E}}).$$

- A derivation of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \& B}\ \textit{With}$$

is mapped to the deduction

$$\dfrac{\begin{matrix}\mathcal{J}(\mathcal{D}_1) \\ \Gamma \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{J}(\mathcal{D}_2) \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \& B}\ (\&_{\mathcal{I}}).$$

- A derivation of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdash A\end{matrix}}{\vdash\, !A}\ \textit{Promotion}$$

is mapped to the deduction

$$\dfrac{\begin{matrix}\mathcal{J}(\mathcal{D}_1) \\ \vdash A\end{matrix}}{\vdash\, !A}\ \textit{Promotion.}$$

4.5 Properties of the translations

As we would expect, we find that the three formulations are actually equivalent. We shall write $\vdash^N \Gamma \overset{\mathcal{D}}{\vdash} A$ to represent a deduction, $\mathcal{D}$, in the natural deduction formulation, $\vdash^S \Gamma \overset{\pi}{\vdash} A$ to represent a proof, π, in the sequent calculus formulation and $\vdash^A \Gamma \overset{\mathcal{D}}{\vdash} A$ to represent a derivation, $\mathcal{D}$, in the axiomatic formulation.

Theorem 7.

- If $\vdash^S \Gamma \overset{\pi}{\vdash} A$ then $\vdash^N \Gamma \overset{\mathcal{N}(\pi)}{\vdash} A$.
- *If* $\vdash^N \Gamma \overset{\mathcal{D}}{\vdash} A$ then $\vdash^S \Gamma \overset{\mathcal{S}(\mathcal{D})}{\vdash} A$.
- If $\vdash^N \Gamma \overset{\mathcal{D}}{\vdash} A$ then $\vdash^A \Gamma \overset{\mathcal{H}(\mathcal{D})}{\vdash} A$.
- If $\vdash^A \Gamma \overset{\mathcal{D}}{\vdash} A$ then $\vdash^N \Gamma \overset{\mathcal{J}(\mathcal{D})}{\vdash} A$.

Proof. By straightforward structural induction. ■

Corollary 1. The natural deduction (Figure 2.3), sequent calculus (Figure 2.1) and axiomatic (Figure 2.4) formulations are equivalent formulations of **ILL**.

5 Translations

As mentioned earlier, the exponential regains the full power of **IL** but in a controlled way. Propositions can be weakened or contracted *provided* they are of the form $!A$. Girard [31, Pages 78–82] showed how **ILL** could be translated to and from **IL**. Here we shall consider the translations in detail but using the natural deduction formulation from §2.[12] First we repeat the natural deduction formulation of **IL** in Figure 2.5.

$$\frac{}{\Gamma, A \vdash A}\, Id$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,(\bot_{\mathcal{E}})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,(\wedge_{\mathcal{I}}) \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,(\wedge_{\mathcal{E}-1}) \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,(\wedge_{\mathcal{E}-2})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,(\supset_{\mathcal{I}}) \qquad \frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\,(\supset_{\mathcal{E}})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,(\vee_{\mathcal{I}-1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,(\vee_{\mathcal{I}-2})$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\,(\vee_{\mathcal{E}})$$

Figure 2.5: Natural Deduction Formulation of **IL**

We shall make a quick observation concerning the formulation of **IL** given in Figure 2.5. We have chosen not to make the structural rules explicit but they are admissible rules, as is the rule of substitution.

Proposition 2. In the formulation of **IL** given in Figure 2.5, the rules

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\, \textit{Weakening}, \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\, \textit{Contraction}$$

$$\text{and} \quad \frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\, \textit{Substitution}$$

are admissible.

Now let us repeat Girard's translation, which is given at the level of propositions.

Definition 11. (Girard)

$$\begin{array}{rcll} \bot^\circ & \stackrel{\text{def}}{=} & \mathbf{f} & \\ A^\circ & \stackrel{\text{def}}{=} & A & \text{If } A \text{ is an atomic formula} \\ (A \wedge B)^\circ & \stackrel{\text{def}}{=} & A^\circ \& B^\circ & \\ (A \vee B)^\circ & \stackrel{\text{def}}{=} & !(A^\circ) \oplus !(B^\circ) & \\ (A \supset B)^\circ & \stackrel{\text{def}}{=} & !(A^\circ) \multimap B^\circ & \end{array}$$

[12] Girard's original presentation used the sequent calculus formulation.

If $\Gamma = A_1, \ldots, \Gamma_n$ then by Γ° we mean $A_1^\circ, \ldots, A_n^\circ$.

Theorem 8. (Girard) If $\vdash_{IL} \Gamma \vdash A$ then $\vdash_{ILL} !(\Gamma^\circ) \vdash A^\circ$.

Proof. By induction on the structure of $\vdash_{IL} \Gamma \vdash A$. We consider the last rule used in the deduction.

- A deduction of the form

$$\frac{}{\Gamma, A \vdash A}\ Id$$

is translated to

$$\dfrac{\dfrac{}{!\Gamma^\circ \vdash !\Gamma^\circ} \qquad \dfrac{\dfrac{}{!A^\circ \vdash !A^\circ}\ \textit{Identity}}{!A^\circ \vdash A^\circ}\ \textit{Dereliction}}{!\Gamma^\circ, !A^\circ \vdash A^\circ}\ \textit{Weakening}^*.$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash \bot\end{array}}{\Gamma \vdash A}\ (\bot_\mathcal{E})$$

is translated to

$$\dfrac{\begin{array}{c}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash \mathbf{f}\end{array}}{!\Gamma^\circ \vdash A^\circ}\ (\mathbf{f}_\mathcal{E}).$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}\ (\wedge_\mathcal{I})$$

is translated to

$$\dfrac{\begin{array}{c}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash A^\circ\end{array} \qquad \begin{array}{c}\mathcal{D}_2^\circ\\ !\Gamma^\circ \vdash B^\circ\end{array}}{!\Gamma^\circ \vdash A^\circ \& B^\circ}\ (\&_\mathcal{I}).$$

- A deduction of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A \wedge B\end{array}}{\Gamma \vdash A}\ (\wedge_{\mathcal{E}-1})$$

is translated to

$$\dfrac{\begin{array}{c}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash A^\circ \& B^\circ\end{array}}{!\Gamma^\circ \vdash A^\circ}\ (\&_{\mathcal{E}-1}).$$

- A deduction of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma \vdash A \wedge B\end{matrix}}{\Gamma \vdash B}\,(\wedge_{\mathcal{E}-2})$$

  is translated to

$$\dfrac{\begin{matrix}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash A^\circ \& B^\circ\end{matrix}}{!\Gamma^\circ \vdash B^\circ}\,(\&_{\mathcal{E}-2}).$$

- A deduction of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \supset B}\,(\supset_{\mathcal{I}})$$

  is translated to

$$\dfrac{\begin{matrix}\mathcal{D}_1^\circ\\ !\Gamma^\circ, !A^\circ \vdash B^\circ\end{matrix}}{!\Gamma^\circ \vdash !A^\circ \multimap B^\circ}\,(\multimap_{\mathcal{I}}).$$

- A deduction of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma \vdash A \supset B\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash B}\,(\supset_{\mathcal{E}})$$

  is translated to

$$\dfrac{\dfrac{}{!\Gamma^\circ \vdash !\Gamma^\circ} \qquad \dfrac{\begin{matrix}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash !A^\circ \multimap B^\circ\end{matrix} \qquad \dfrac{\dfrac{}{!\Gamma^\circ \vdash !\Gamma^\circ} \qquad \begin{matrix}\mathcal{D}_2^\circ\\ !\Gamma^\circ \vdash A^\circ\end{matrix}}{!\Gamma^\circ \vdash !A^\circ}\,\mathit{Promotion}}{!\Gamma^\circ, !\Gamma^\circ \vdash B^\circ}\,(\multimap_{\mathcal{E}})}{!\Gamma^\circ \vdash B^\circ}\,\mathit{Contraction}^*.$$

- A deduction of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash A \vee B}\,(\vee_{\mathcal{I}-1})$$

  is translated to

$$\dfrac{\dfrac{\dfrac{}{!\Gamma^\circ \vdash !\Gamma^\circ} \qquad \begin{matrix}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash A^\circ\end{matrix}}{!\Gamma^\circ \vdash !A^\circ}\,\mathit{Promotion}}{!\Gamma^\circ \vdash !A^\circ \oplus !B^\circ}\,(\oplus_{\mathcal{I}-1}).$$

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \vee B}(\vee_{\mathcal{I}-2})$$

is translated to

$$\frac{\dfrac{\overline{!\Gamma^\circ \vdash !\Gamma^\circ} \qquad \begin{array}{c}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash B^\circ\end{array}}{!\Gamma^\circ \vdash !B^\circ}\textit{Promotion}}{!\Gamma^\circ \vdash !A^\circ \oplus !B^\circ}(\oplus_{\mathcal{I}-2}).$$

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A \vee B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A \vdash C\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}(\vee_{\mathcal{E}})$$

is translated to

$$\frac{\overline{!\Gamma^\circ \vdash !\Gamma^\circ} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1^\circ\\ !\Gamma^\circ \vdash !A^\circ \oplus !B^\circ\end{array} \qquad \begin{array}{c}\mathcal{D}_2^\circ\\ !\Gamma^\circ, !A^\circ \vdash C^\circ\end{array} \qquad \begin{array}{c}\mathcal{D}_3^\circ\\ !\Gamma^\circ, !B^\circ \vdash C^\circ\end{array}}{!\Gamma^\circ, !\Gamma^\circ \vdash C^\circ}(\oplus_{\mathcal{E}})}{!\Gamma^\circ \vdash C^\circ}\textit{Contraction}^*.$$

■

We can also define the translation of the admissible rules given in Proposition 2.

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash B\end{array}}{\Gamma, A \vdash B}\textit{Weakening}$$

is translated to

$$\frac{\dfrac{}{!A^\circ \vdash !A^\circ}\textit{Identity} \qquad \begin{array}{c}\mathcal{D}_1^\circ\\ !(\Gamma^\circ) \vdash B^\circ\end{array}}{!(\Gamma^\circ), !A^\circ \vdash B^\circ}\textit{Weakening}.$$

- A deduction of the form

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\textit{Contraction}$$

is translated to

$$\frac{\dfrac{}{!A^\circ \vdash !A^\circ}\textit{Identity} \qquad \begin{array}{c}\mathcal{D}_1\\ !(\Gamma^\circ), !A^\circ, !A^\circ \vdash B^\circ\end{array}}{!(\Gamma^\circ), !A^\circ \vdash B^\circ}\textit{Contraction}.$$

- A deduction of the form

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash B}\,\textit{Substitution}$$

is translated to

$$\frac{\dfrac{}{!(\Gamma^\circ) \vdash\, !(\Gamma^\circ)}\,\textit{Id.} \qquad \dfrac{\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash\, !(\Gamma^\circ)}\,\textit{Id.} \qquad \begin{array}{c}\mathcal{D}_1^\circ\\ !(\Gamma^\circ) \vdash A^\circ\end{array}}{!(\Gamma^\circ) \vdash\, !A^\circ}\,\textit{Prom.} \qquad \begin{array}{c}\mathcal{D}_2^\circ\\ !(\Gamma^\circ), !A^\circ \vdash B^\circ\end{array}}{!(\Gamma^\circ), !(\Gamma^\circ) \vdash B^\circ}\,\textit{Subs}}{!(\Gamma^\circ) \vdash B^\circ}\,\textit{Cont}^*.$$

We find that Girard's translation is well behaved with respect to reduction.

Theorem 9. If $\vdash_{IL} \Gamma \overset{\mathcal{D}_1}{\vdash} A \leadsto_\beta \Gamma \overset{\mathcal{D}_1'}{\vdash} A$ then $\vdash_{ILL} !(\Gamma^\circ) \overset{\mathcal{D}_1^\circ}{\vdash} A^\circ \leadsto_\beta^+ !(\Gamma^\circ) \overset{\mathcal{D}_1'^\circ}{\vdash} A^\circ$.

Proof. By induction on the structure of $\Gamma \overset{\mathcal{D}_1}{\vdash} A$. We shall just give three cases.

- The deduction

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}\,(\wedge_{\mathcal{I}})}{\Gamma \vdash A}\,(\wedge_{\mathcal{E}-1})$$

which reduces to the following deduction, $\mathcal{D}$

$$\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array}$$

is translated to the following deduction

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1^\circ\\ !(\Gamma^\circ) \vdash A^\circ\end{array} \qquad \begin{array}{c}\mathcal{D}_2^\circ\\ !(\Gamma^\circ) \vdash B^\circ\end{array}}{!(\Gamma^\circ) \vdash A^\circ \& B^\circ}\,(\&_{\mathcal{I}})}{!(\Gamma^\circ) \vdash A^\circ}\,(\&_{\mathcal{E}-1})$$

which reduces to

$$\begin{array}{c}\mathcal{D}_1^\circ\\ !(\Gamma^\circ) \vdash A^\circ\end{array}$$

which is equal to $\mathcal{D}^\circ$.

- The deduction

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \vee B}\,(\vee_{\mathcal{I}-1}) \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma, A \vdash C\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\,(\vee_{\mathcal{E}})$$

which reduces to the following deduction, $\mathcal{D}$

$$\dfrac{\overset{\mathcal{D}_1}{\Gamma \vdash A} \qquad \overset{\mathcal{D}_2}{\Gamma, A \vdash C}}{\Gamma \vdash C}\ \textit{Substitution}$$

is translated to the following deduction

$$\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id.} \qquad \overset{\mathcal{D}_1^\circ}{!(\Gamma^\circ) \vdash A^\circ}}{!(\Gamma^\circ) \vdash !A^\circ}\ \textit{Prom}}{!(\Gamma^\circ) \vdash !A^\circ \oplus !B^\circ}\ (\oplus_{\mathcal{I}-1}) \qquad \overset{\mathcal{D}_2^\circ}{!(\Gamma^\circ), !A^\circ \vdash C^\circ} \qquad \overset{\mathcal{D}_3^\circ}{!(\Gamma^\circ), !B^\circ \vdash C^\circ}}{!(\Gamma^\circ), !(\Gamma^\circ) \vdash C^\circ}\ (\oplus_{\mathcal{E}})}{!(\Gamma^\circ) \vdash C^\circ}\ \textit{Contr.}^*$$

which reduces to

$$\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id.} \qquad \dfrac{\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id.} \qquad \overset{\mathcal{D}_1^\circ}{!(\Gamma^\circ) \vdash A^\circ}}{!(\Gamma^\circ) \vdash !A^\circ}\ \textit{Prom.} \qquad \overset{\mathcal{D}_2^\circ}{!(\Gamma^\circ), !A^\circ \vdash C^\circ}}{!(\Gamma^\circ), !(\Gamma^\circ) \vdash C^\circ}\ \textit{Subs.}}{!(\Gamma^\circ) \vdash C^\circ}\ \textit{Cont.}^*$$

which is equal to $\mathcal{D}^\circ$.

- The deduction

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma, A \vdash B}}{\Gamma \vdash A \supset B}\ (\supset_{\mathcal{I}}) \qquad \overset{\mathcal{D}_2}{\Gamma \vdash A}}{\Gamma \vdash B}\ (\supset_{\mathcal{E}})$$

which reduces to the following deduction, $\mathcal{D}$

$$\dfrac{\overset{\mathcal{D}_2}{\Gamma \vdash A} \qquad \overset{\mathcal{D}_1}{\Gamma, A \vdash B}}{\Gamma \vdash B}\ \textit{Substitution}$$

is translated to the following deduction

$$\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id.} \qquad \dfrac{\dfrac{\overset{\mathcal{D}_1^\circ}{!(\Gamma^\circ), !A^\circ \vdash B^\circ}}{!(\Gamma^\circ) \vdash !A^\circ \multimap B^\circ}\ (\multimap_{\mathcal{I}}) \qquad \dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id.} \qquad \overset{\mathcal{D}_2^\circ}{!(\Gamma^\circ) \vdash A^\circ}}{!(\Gamma^\circ) \vdash !A^\circ}\ \textit{Prom.}}{!(\Gamma^\circ), !(\Gamma^\circ) \vdash B^\circ}\ (\multimap_{\mathcal{E}})}{!(\Gamma^\circ) \vdash B^\circ}\ \textit{Contr.}^*$$

which reduces to

$$\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id.} \qquad \dfrac{\dfrac{\dfrac{}{!(\Gamma^\circ) \vdash !(\Gamma^\circ)}\ \textit{Id.} \qquad \overset{\mathcal{D}_2^\circ}{!(\Gamma^\circ) \vdash A^\circ}}{!(\Gamma^\circ) \vdash !A^\circ}\ \textit{Prom.} \qquad \overset{\mathcal{D}_1^\circ}{!(\Gamma^\circ), !A^\circ \vdash B^\circ}}{!(\Gamma^\circ), !(\Gamma^\circ) \vdash B^\circ}\ \textit{Subs.}}{!(\Gamma^\circ) \vdash B^\circ}\ \textit{Contr.}^*$$

which is equal to $\mathcal{D}^\circ$.

■

Theorem 10. If $\vdash_{IL} \Gamma \vdash A$ is in β-normal form, then so is $\vdash_{ILL} !(\Gamma^{\circ}) \vdash A^{\circ}$.

The above theorem is *not* true if we consider (β, c)-normal form and we shall consider this point further in the next chapter. There are many alternatives to the classic Girard translation. For example that given below, in Definition 12, avoids the use of the With connective in the translation of the conjunction. Thus for the $(\supset, \wedge)$-fragment of **IL**, it is a purely multiplicative translation.

Definition 12.

$$\begin{array}{rcll}
\bot^{\star} & \stackrel{\text{def}}{=} & \mathbf{f} & \\
A^{\star} & \stackrel{\text{def}}{=} & A & \text{If } A \text{ is an atomic formula} \\
(A \wedge B)^{\star} & \stackrel{\text{def}}{=} & !(A^{\star}) \otimes !(B^{\star}) & \\
(A \vee B)^{\star} & \stackrel{\text{def}}{=} & !(A^{\star}) \oplus !(B^{\star}) & \\
(A \supset B)^{\star} & \stackrel{\text{def}}{=} & !(A^{\star}) \multimap B^{\star} &
\end{array}$$

Theorem 11. If $\vdash_{IL} \Gamma \vdash A$ then $\vdash_{ILL} !\Gamma^{\star} \vdash A^{\star}$.

It is trivial to see that there is a translation in the opposite direction, from **ILL** to **IL**.

Definition 13.

$$\begin{array}{rcll}
\mathbf{t}^{\mathsf{s}} & \stackrel{\text{def}}{=} & \top & \\
\mathbf{f}^{\mathsf{s}} & \stackrel{\text{def}}{=} & \bot & \\
A^{\mathsf{s}} & \stackrel{\text{def}}{=} & A & \text{If } A \text{ is an atomic formula} \\
(A \otimes B)^{\mathsf{s}} & \stackrel{\text{def}}{=} & A^{\mathsf{s}} \wedge B^{\mathsf{s}} & \\
(A \& B)^{\mathsf{s}} & \stackrel{\text{def}}{=} & A^{\mathsf{s}} \wedge B^{\mathsf{s}} & \\
(A \oplus B)^{\mathsf{s}} & \stackrel{\text{def}}{=} & A^{\mathsf{s}} \vee B^{\mathsf{s}} & \\
(A \multimap B)^{\mathsf{s}} & \stackrel{\text{def}}{=} & A^{\mathsf{s}} \supset B^{\mathsf{s}} & \\
(!A)^{\mathsf{s}} & \stackrel{\text{def}}{=} & A^{\mathsf{s}} &
\end{array}$$

Theorem 12. If $\vdash_{ILL} \Gamma \vdash A$ then $\vdash_{IL} \Gamma^{\mathsf{s}} \vdash A^{\mathsf{s}}$.

Proof. By induction on the structure of the deduction, $\Gamma \vdash A$. The *Promotion* rule becomes a series of *Substitution* rules and the *Dereliction* rule becomes a dummy rule. ■

We can also present Girard's translation at the level of an axiomatic formulation. First let us recall an axiomatic formulation of **IL** in Figure 2.6.

Theorem 13. If $\vdash_{IL} \Gamma \vdash A$ then $\vdash_{ILL} !(\Gamma^{\circ}) \vdash A^{\circ}$.

Proof. By induction on the structure of $\vdash_{IL} \Gamma \vdash A$. (Omitted). ■

$$
\begin{array}{rcl}
\texttt{S} & : & (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \\
\texttt{K} & : & A \supset (B \supset A) \\
\\
\texttt{and} & : & A \supset (B \supset (A \wedge B)) \\
\texttt{proj}_1 & : & A \wedge B \supset A \\
\texttt{proj}_2 & : & A \wedge B \supset B \\
\\
\texttt{inleft} & : & A \supset A \vee B \\
\texttt{inright} & : & B \supset A \vee B \\
\texttt{or} & : & (A \supset B) \supset ((C \supset B) \supset (A \vee B \supset C)) \\
\\
\texttt{false} & : & \bot \supset A
\end{array}
$$

$$
\frac{}{A \vdash A}\ \mathit{Identity} \qquad \frac{}{\vdash A}\ \mathit{Axiom}
$$

$$
\frac{\Gamma \vdash A \supset B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ \mathit{Modus\ Ponens}
$$

Figure 2.6: Axiomatic Formulation of **IL**

Chapter 3
Term Assignment

1 The Curry-Howard Correspondence

As explained in Chapter 1 we can use the so-called Curry-Howard correspondence [41] to derive a term assignment system for **ILL** given a natural deduction formulation. The correspondence essentially annotates a derivation with a *term* which can be viewed as an encoding of the derivation. In particular, given the term and context one should be able to reconstruct the derivation. An alternative view is that the correspondence provides a type system for a term calculus. In addition, the correspondence also links term reduction to proof normalization. We shall utilize this in §4. First let us use the correspondence to derive a term assignment system, which is given in Figure 3.1.

Thus a type, A, is given by the same grammar as for a formula in Chapter 2 and a linear term, M, is given by the grammar

$$\begin{array}{rcl} M & ::= & x \mid \mathsf{true}(\vec{M}) \mid \mathsf{false}_A(\vec{M}\,;\,M) \mid \\ & & \lambda x\colon A.M \mid MM \mid * \mid \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ M \mid \\ & & M\otimes M \mid \mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ M \mid \langle M,M\rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \\ & & \mathsf{inl}(M) \mid \mathsf{inr}(M) \mid \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \rightarrow M \parallel \mathsf{inr}(y) \rightarrow M \mid \\ & & \mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M \mid \mathsf{derelict}(M) \mid \\ & & \mathsf{discard}\ M\ \mathsf{in}\ M \mid \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ M. \end{array}$$

Henceforth, however, we shall only be interested in *well-typed* terms, i.e. those for which there exists a context, Γ, and type A, such that $\Gamma \triangleright M\colon A$ holds given the rules in Figure 3.1.

Often we shall use some shorthand notation for the terms. We shall often use a vector notation $\vec{M}$ to denote a sequence of terms, for example, $\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N$ in place of $\mathsf{promote}\ M_1,\ldots,M_n\ \mathsf{for}\ x_1,\ldots,\ldots,x_n\ \mathsf{in}\ N$, where it is not necessary to identify the subterms. We shall also write compound terms for brevity; for example, we shall write $\mathsf{discard}\ M_1,\ldots,M_n\ \mathsf{in}\ N$ to denote the (larger) term $\mathsf{discard}\ M_1\ \mathsf{in}\ \ldots \mathsf{discard}\ M_n\ \mathsf{in}\ N$. Now let us consider a few properties of the term assignment system. Firstly, we notice that for the multiplicative, exponential fragment, we have the required linearity property that a free variable occurs exactly once within a term. First let us fix the definition of a free variable.

Definition 14. The set $FV(M)$ of *free variables* of a linear term M is defined inductively as follows.

$$\begin{array}{rcl} FV(x) & \stackrel{\text{def}}{=} & \{x\} \\ FV(\lambda x\colon A.M) & \stackrel{\text{def}}{=} & FV(M) - \{x\} \\ FV(MN) & \stackrel{\text{def}}{=} & FV(M) \cup FV(N) \\ FV(*) & \stackrel{\text{def}}{=} & \emptyset \\ FV(\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N) & \stackrel{\text{def}}{=} & FV(M) \cup FV(N) \\ FV(M\otimes N) & \stackrel{\text{def}}{=} & FV(M) \cup FV(N) \\ FV(\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ N) & \stackrel{\text{def}}{=} & FV(M) \cup (FV(N) - \{x,y\}) \\ FV(\langle M,N\rangle) & \stackrel{\text{def}}{=} & FV(M) \qquad (= FV(N)) \\ FV(\mathsf{fst}(M)) & \stackrel{\text{def}}{=} & FV(M) \\ FV(\mathsf{snd}(M)) & \stackrel{\text{def}}{=} & FV(M) \end{array}$$

$$\frac{}{x\colon A \triangleright x\colon A}\ \textit{Identity}$$

$$\frac{\Gamma_1 \triangleright M_1\colon A_1 \quad \cdots \quad \Gamma_n \triangleright M_n\colon A_n}{\Gamma_1, \ldots, \Gamma_n \triangleright \mathsf{true}(\vec{M})\colon \mathbf{t}}\ (\mathbf{t}_\mathcal{I}) \qquad \frac{\Gamma_1 \triangleright M_1\colon A_1 \quad \cdots \quad \Gamma_n \triangleright M_n\colon A_n \qquad \Delta \triangleright N\colon \mathbf{f}}{\Gamma_1, \ldots, \Gamma_n, \Delta \triangleright \mathsf{false}_B(\vec{M}\ ;\ N)\colon B}\ (\mathbf{f}_\mathcal{E})$$

$$\frac{\Gamma, x\colon A \triangleright M\colon B}{\Gamma \triangleright \lambda x\colon A.M\colon A \multimap B}\ (\multimap_\mathcal{I}) \qquad \frac{\Gamma \triangleright M\colon A \multimap B \qquad \Delta \triangleright N\colon A}{\Gamma, \Delta \triangleright MN\colon B}\ (\multimap_\mathcal{E})$$

$$\frac{}{\triangleright *\colon I}\ (I_\mathcal{I}) \qquad \frac{\Delta \triangleright N\colon I \qquad \Gamma \triangleright M\colon A}{\Gamma, \Delta \triangleright \mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ N\colon A}\ (I_\mathcal{E})$$

$$\frac{\Gamma \triangleright M\colon A \qquad \Delta \triangleright N\colon B}{\Gamma, \Delta \triangleright M \otimes N\colon A \otimes B}\ (\otimes_\mathcal{I}) \qquad \frac{\Delta \triangleright M\colon A \otimes B \qquad \Gamma, x\colon A, y\colon B \triangleright N\colon C}{\Gamma, \Delta \triangleright \mathsf{let}\ M\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ N\colon C}\ (\otimes_\mathcal{E})$$

$$\frac{\Gamma \triangleright M\colon A \qquad \Gamma \triangleright N\colon B}{\Gamma \triangleright \langle M, N\rangle\colon A \& B}\ (\&_\mathcal{I}) \qquad \frac{\Gamma \triangleright M\colon A \& B}{\Gamma \triangleright \mathsf{fst}(M)\colon A}\ (\&_{\mathcal{E}-1}) \qquad \frac{\Gamma \triangleright M\colon A \& B}{\Gamma \triangleright \mathsf{snd}(M)\colon B}\ (\&_{\mathcal{E}-2})$$

$$\frac{\Gamma \triangleright M\colon A}{\Gamma \triangleright \mathsf{inl}(M)\colon A \oplus B}\ (\oplus_{\mathcal{I}-1}) \qquad \frac{\Gamma \triangleright M\colon B}{\Gamma \triangleright \mathsf{inr}(M)\colon A \oplus B}\ (\oplus_{\mathcal{I}-2})$$

$$\frac{\Delta \triangleright M\colon A \oplus B \qquad \Gamma, x\colon A \triangleright N\colon C \qquad \Gamma, y\colon B \triangleright P\colon C}{\Gamma, \Delta \triangleright \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P\colon C}\ (\oplus_\mathcal{E})$$

$$\frac{\Gamma_1 \triangleright M_1\colon !A_1 \quad \cdots \quad \Gamma_n \triangleright M_n\colon !A_n \qquad x_1\colon !A_1, \ldots, x_n\colon !A_n \triangleright N\colon B}{\Gamma_1, \ldots, \Gamma_n \triangleright \mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N\colon !B}\ \textit{Promotion}$$

$$\frac{\Gamma \triangleright M\colon !A \qquad \Delta \triangleright N\colon B}{\Gamma, \Delta \triangleright \mathsf{discard}\ M\ \mathsf{in}\ N\colon B}\ \textit{Weakening}$$

$$\frac{\Delta \triangleright M\colon !A \qquad \Gamma, x\colon !A, y\colon !A \triangleright N\colon B}{\Gamma, \Delta \triangleright \mathsf{copy}\ M\ \mathsf{as}\ x, y\ \mathsf{in}\ N\colon B}\ \textit{Contraction}$$

$$\frac{\Gamma \triangleright M\colon !A}{\Gamma \triangleright \mathsf{derelict}(M)\colon A}\ \textit{Dereliction}$$

Figure 3.1: Term Assignment for Natural Deduction Formulation of **ILL**.

$$
\begin{aligned}
FV(\mathsf{inl}(M)) &\stackrel{\text{def}}{=} FV(M)\\
FV(\mathsf{inr}(M)) &\stackrel{\text{def}}{=} FV(M)\\
FV(\mathsf{true}(\vec{M})) &\stackrel{\text{def}}{=} \textstyle\bigcup_{i=1}^{n} FV(M_i)\\
FV(\mathsf{false}_A(\vec{M}\ ;\ N)) &\stackrel{\text{def}}{=} (\textstyle\bigcup_{i=1}^{n} FV(M_i)) \cup FV(N)\\
FV(\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \rightarrow N \parallel \mathsf{inr}(y) \rightarrow P) &\stackrel{\text{def}}{=} FV(M) \cup (FV(N) - \{x\})\\
FV(\mathsf{promote}\ M_1, \ldots M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N) &\stackrel{\text{def}}{=} \textstyle\bigcup_{i=1}^{n} FV(M_i) \quad \text{where } x \in FV(M_i) \text{ say}\\
FV(\mathsf{discard}\ M\ \mathsf{in}\ N) &\stackrel{\text{def}}{=} FV(M) \cup FV(N)\\
FV(\mathsf{copy}\ M\ \mathsf{as}\ x, y\ \mathsf{in}\ N) &\stackrel{\text{def}}{=} FV(M) \cup (FV(N) - \{x, y\})\\
FV(\mathsf{derelict}(M)) &\stackrel{\text{def}}{=} FV(M)
\end{aligned}
$$

We say that a term M is *closed* if $FV(M) = \emptyset$. We can define the following property of the linear term calculus.

Proposition 3. If M is a valid term from the multiplicative, exponential fragment, then $\forall x \in FV(M)$, x occurs exactly once in M.

This means that for the multiplicative, exponential fragment we have completely *linear* substitution. We can define substitution as follows.

Definition 15. Given two terms M and N (where $\vec{w}\colon \Gamma \triangleright M\colon A$ and $\Delta, x\colon A \triangleright N\colon B$), we can define $N[x := M]$, the substitution of M for x in N, by induction on the structure of N.[1]

$$
\begin{aligned}
x[x := M] &\stackrel{\text{def}}{=} M\\
(\lambda y\colon A.N)[x := M] &\stackrel{\text{def}}{=} \lambda y\colon A.(N[x := M])\\
(NP)[x := M] &\stackrel{\text{def}}{=} \begin{cases} (N[x := M])P & \text{If } x \in FV(N)\\ N(P[x := M]) & \text{Otherwise}\end{cases}
\end{aligned}
$$

$$
(\mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ P)[x := M] \stackrel{\text{def}}{=} \begin{cases} \mathsf{let}\ (N[x := M])\ \mathsf{be}\ *\ \mathsf{in}\ P & \text{If } x \in FV(N)\\ \mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ P[x := M] & \text{Otherwise}\end{cases}
$$

$$
(N \otimes P)[x := M] \stackrel{\text{def}}{=} \begin{cases} (N[x := M]) \otimes P & \text{If } x \in FV(N)\\ N \otimes (P[x := M]) & \text{Otherwise}\end{cases}
$$

$$
(\mathsf{let}\ N\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ P)[x := M] \stackrel{\text{def}}{=} \begin{cases} \mathsf{let}\ (N[x := M])\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ P & \text{If } x \in FV(N)\\ \mathsf{let}\ N\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ (P[x := M]) & \text{Otherwise}\end{cases}
$$

$$
\begin{aligned}
(\langle N, P\rangle)[x := M] &\stackrel{\text{def}}{=} \langle (N[x := M]), (P[x := M])\rangle\\
(\mathsf{fst}(N))[x := M] &\stackrel{\text{def}}{=} \mathsf{fst}(N[x := M])\\
(\mathsf{snd}(N))[x := M] &\stackrel{\text{def}}{=} \mathsf{snd}(N[x := M])\\
(\mathsf{inl}(N))[x := M] &\stackrel{\text{def}}{=} \mathsf{inl}(N[x := M])\\
(\mathsf{inr}(N))[x := M] &\stackrel{\text{def}}{=} \mathsf{inr}(N[x := M])
\end{aligned}
$$

[1]We shall assume that the terms have been suitably α-converted so that variable capture will not occur.

$$(\mathsf{false}_A(\vec{N}\,;\,P))[x:=M] \stackrel{\text{def}}{=} \begin{cases} \mathsf{false}_A(\vec{N}\,;\,P[x:=M]) & \text{If } x \in FV(P) \\ \mathsf{false}_A(N_1,\ldots,N_i[x:=M],\ldots,N_n\,;\,P) & \text{Otherwise} \end{cases}$$

$$(\mathsf{true}(\vec{N}))[x:=M] \stackrel{\text{def}}{=} \mathsf{true}(N_1,\ldots,N_i[x:=M],\ldots,N_n)$$

$$\begin{array}{l}(\mathsf{case}\ N\ \mathsf{of} \\ \quad \mathsf{inl}(y) \to P \\ \quad \mathsf{inr}(z) \to Q)[x:=M]\end{array} \stackrel{\text{def}}{=} \begin{cases} \begin{array}{l}\mathsf{case}\ (N[x:=M])\ \mathsf{of} \\ \quad \mathsf{inl}(y) \to P \\ \quad \mathsf{inr}(z) \to Q\end{array} & \text{If } x \in FV(N) \\ \begin{array}{l}\mathsf{case}\ N\ \mathsf{of} \\ \quad \mathsf{inl}(x) \to (P[x:=M]) \\ \quad \mathsf{inr}(z) \to (Q[x:=M])\end{array} & \text{Otherwise} \end{cases}$$

$$(\mathsf{promote}\ \vec{N}\ \mathsf{for}\ \vec{y}\ \mathsf{in}\ P)[x:=M] \stackrel{\text{def}}{=} \mathsf{promote}\ N_1,\ldots,(N_i[x:=M]),\ldots,N_n\ \mathsf{for}\ \vec{y}\ \mathsf{in}\ P$$

$$(\mathsf{discard}\ N\ \mathsf{in}\ P)[x:=M] \stackrel{\text{def}}{=} \begin{cases} \mathsf{discard}\ (N[x:=M])\ \mathsf{in}\ P & \text{If } x \in FV(N) \\ \mathsf{discard}\ N\ \mathsf{in}\ (P[x:=M]) & \text{Otherwise} \end{cases}$$

$$(\mathsf{copy}\ N\ \mathsf{as}\ y,z\ \mathsf{in}\ P)[x:=M] \stackrel{\text{def}}{=} \begin{cases} \mathsf{copy}\ (N[x:=M])\ \mathsf{as}\ x,y\ \mathsf{in}\ P & \text{If } x \in FV(N) \\ \mathsf{copy}\ N\ \mathsf{as}\ y,z\ \mathsf{in}\ (P[x:=M]) & \text{Otherwise} \end{cases}$$

Using this definition we can see that substitution is well-defined in terms of the syntax. This property is known as *closure under substitution* and its importance is obvious. Wadler [79] first showed that previous proposals for the linear term calculus [1, 57, 55, 75] did not have this property. As explained in Chapter 2, the formulation here has been chosen with this important property in mind.

Lemma 3. If $\Delta \triangleright N\colon A$ and $\Gamma, x\colon A \triangleright M\colon B$ then $\Gamma, \Delta \triangleright M[x:=N]\colon B$.

Proof. By induction on $\Gamma, x\colon A \triangleright M\colon B$. ■

Finally let us return to the considerations of Chapter 2 where we suggested an alternative formulation for the additive connectives. For example, we proposed the following alternative formulation for the With introduction rule

$$\frac{\begin{array}{c}\vdots \\ A_1\end{array} \;\cdots\; \begin{array}{c}\vdots \\ A_n\end{array} \qquad \begin{array}{c}[\![A_1^{x_1} \;\cdots\; A_n^{x_n}]\!] \\ \vdots \\ B\end{array} \qquad \begin{array}{c}[\![A_1^{y_1} \;\cdots\; A_n^{y_n}]\!] \\ \vdots \\ C\end{array}}{B\&C}\,(\&_{\mathcal{I}})_{x1,\ldots,xn,y1,\ldots,yn}.$$

This would mean a rule at the level of terms of the following form (where we revert to the presentation in sequent form)

$$\frac{\Gamma_1 \triangleright M_1\colon A_1 \quad \cdots \quad \Gamma_n \triangleright M_n\colon A_n \qquad \begin{array}{c} x_1\colon A_1,\ldots,x_n\colon A_n \triangleright N\colon B \\ y_1\colon A_1,\ldots,y_n\colon A_n \triangleright P\colon C\end{array}}{\Gamma_1,\ldots,\Gamma_n \triangleright \langle N,P\rangle\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \langle \vec{x},\vec{y}\rangle\colon B\&C}\,(\mathit{With}_{\mathcal{I}}).$$

(As mentioned earlier, this formulation is similar to that proposed by Girard [31] for the proof net formulation of classical linear logic. Indeed it should be compared to the term syntax for proof nets proposed by Abramsky [1].) As we can see, our chosen formulation is less verbose at the cost of ensuring that the variables used in both terms are exactly the same.

In fact, we could propose yet another formulation which is a combination of that given above and that which we use in Figure 3.1

$$\frac{\Gamma_1 \triangleright M_1\colon A_1 \quad \cdots \quad \Gamma_n \triangleright M_n\colon A_n \qquad \begin{array}{c} x_1\colon A_1,\ldots,x_n\colon A_n \triangleright N\colon B \\ y_1\colon A_1,\ldots,y_n\colon A_n \triangleright P\colon C\end{array}}{\Gamma_1,\ldots,\Gamma_n \triangleright \langle N[\vec{x}:=\vec{z}],P[\vec{y}:=\vec{z}]\rangle\ \mathsf{with}\ \vec{M}\ \mathsf{for}\ \vec{z}\colon B\&C}\,(\mathit{With}'_{\mathcal{I}}).$$

This version allows for multiplicative contexts which are then α-converted so as to be equal.

1.1 Comparison with Existing Syntax

We should point out that the syntax chosen for the linear term calculus is slightly different to that of existing functional programming languages such as Haskell [42]. There the syntax for pattern matching is of the form

$$\texttt{let}\ \mathit{patt} = \mathit{exp}\ \texttt{in}\ \mathit{exp},$$

rather than that of the linear term calculus

$$\mathsf{let}\ \mathit{exp}\ \mathsf{be}\ \mathit{patt}\ \mathsf{in}\ \mathit{exp}.$$

The syntax here was chosen to be similar to those already existing for linear logic [1, 57]. Also the `let` construct in Haskell introduces *polymorphic* definitions, and given that this thesis does not address type quantification, we keep our syntax to avoid any confusion.

1.2 An Additional Equality

We have pointed out already that we do not introduce syntax for applications of the *Exchange* rule. However, the syntax for the *Promotion* rule as it stands can distinguish between proofs which differ just by applications of the *Exchange* rule. This is essentially due to inherently linear nature of syntax. For example, consider the terms

$$\mathsf{promote}\ M_1, M_2\ \mathsf{for}\ x_1, x_2\ \mathsf{in}\ N, \quad \text{and} \quad \mathsf{promote}\ M_2, M_1\ \mathsf{for}\ x_2, x_1\ \mathsf{in}\ N.$$

Clearly we should want these two terms to be considered equal. Thus we shall implicitly assume the following equality for the rest of the thesis

$$\mathsf{promote}\ \vec{M}, M_1, M_2, \vec{M'}\ \mathsf{for}\ \vec{x}, y_1, y_2, \vec{z}\ \mathsf{in}\ N \quad =_{ex} \quad \mathsf{promote}\ \vec{M}, M_2, M_1, \vec{M'}\ \mathsf{for}\ \vec{x}, y_2, y_1, \vec{z}\ \mathsf{in}\ N.$$

2 Term Assignment for Sequent Calculus

Unfortunately, as discussed in Chapter 1 there is no real equivalent to the Curry-Howard correspondence for the sequent calculus formulation. This is because many different sequent derivations denote the same logical deduction (i.e. the $\mathcal{N}$ mapping from Chapter 2 is a many-to-one mapping). To derive a term assignment system for the sequent calculus formulation, we have two options.

1. To use categorical considerations to suggest a syntax. This technique is used by Benton *et al.* [14] for the multiplicative, exponential fragment of **ILL**.
2. To use the procedures for mapping proofs between the sequent calculus and natural deduction formulations.

Of course, both these techniques should converge to the same answer! For conciseness we shall not detail the second option, but simply consider an example. Consider the (sequent calculus formulation of the) *Promotion* rule

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \mathit{Promotion}.$$

Let us assume that the upper sequent is represented by the term M, thus $\vec{x}\colon !\Gamma \triangleright M\colon A$. Using the $\mathcal{N}$ procedure to map this proof into a deduction in the natural deduction formulation we get

$$\frac{\overline{\vec{y}\colon !\Gamma \triangleright \vec{y}\colon !\Gamma} \qquad \vec{x}\colon !\Gamma \triangleright \mathcal{N}(M)\colon A}{\vec{y} \triangleright \mathsf{promote}\ \vec{y}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ \mathcal{N}(M)\colon !A}\ \mathit{Promotion}.$$

Thus we can conclude that the term assignment for the *Promotion* rule in the sequent calculus formulation is of the form

$$\frac{}{x\colon A \rhd x\colon A}\ \textit{Identity}$$

$$\frac{\Gamma \rhd M\colon B \qquad x\colon B, \Delta \rhd N\colon C}{\Gamma, \Delta \rhd N[x := M]\colon C}\ \textit{Cut}$$

$$\frac{}{\vec{x}\colon \Gamma, y\colon \mathbf{f} \rhd \mathsf{false}_A(\vec{x}\,;\, y)\colon A}\ (\mathbf{f}_{\mathcal{L}}) \qquad \frac{}{\vec{x}\colon \Gamma \rhd \mathsf{true}(\vec{x})\colon \mathbf{t}}\ (\mathbf{t}_{\mathcal{R}})$$

$$\frac{\Gamma \rhd M\colon A}{\Gamma, x\colon I \rhd \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ M\colon A}\ (I_{\mathcal{L}}) \qquad \frac{}{\rhd *\colon I}\ (I_{\mathcal{R}})$$

$$\frac{\Gamma, x\colon A, y\colon B \rhd M\colon C}{\Gamma, z\colon A\otimes B \rhd \mathsf{let}\ z\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ M\colon C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon B}{\Gamma, \Delta \rhd M\otimes N\colon A\otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \rhd M\colon A \qquad \Delta, x\colon B \rhd N\colon C}{\Gamma, \Delta, y\colon A\multimap B \rhd N[x := (yM)]\colon C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, x\colon A \rhd M\colon B}{\Gamma \rhd \lambda x\colon A.M\colon A\multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma, x\colon A \rhd M\colon C}{\Gamma, y\colon A\& B \rhd M[x := \mathsf{fst}(y)]\colon C}\ (\&_{\mathcal{L}-1}) \qquad \frac{\Gamma, x\colon B \rhd M\colon C}{\Gamma, y\colon A\& B \rhd M[x := \mathsf{snd}(y)]\colon C}\ (\&_{\mathcal{L}-2})$$

$$\frac{\vec{x}\colon \Gamma \rhd M\colon A \qquad \vec{x}\colon \Gamma \rhd N\colon B}{\vec{x}\colon \Gamma \rhd \langle M, N\rangle\colon A\& B}\ (\&_{\mathcal{R}})$$

$$\frac{\vec{x}\colon \Gamma, y\colon A \rhd M\colon C \qquad \vec{x}\colon \Gamma, z\colon B \rhd N\colon C}{\vec{x}\colon \Gamma, w\colon A\oplus B \rhd \mathsf{case}\ w\ \mathsf{of}\ \mathsf{inl}(y)\ \to\ M \parallel \mathsf{inr}(z)\ \to\ N\colon C}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Gamma \rhd M\colon A}{\Gamma \rhd \mathsf{inl}(M)\colon A\oplus B}\ (\oplus_{\mathcal{R}-1}) \qquad \frac{\Gamma \rhd M\colon B}{\Gamma \rhd \mathsf{inr}(M)\colon A\oplus B}\ (\oplus_{\mathcal{R}-2})$$

$$\frac{\Gamma \rhd M\colon B}{\Gamma, x\colon !A \rhd \mathsf{discard}\ x\ \mathsf{in}\ M\colon B}\ \textit{Weakening} \qquad \frac{\Gamma, y\colon !A, z\colon !A \rhd M\colon B}{\Gamma, x\colon !A \rhd \mathsf{copy}\ x\ \mathsf{as}\ y, z\ \mathsf{in}\ M\colon B}\ \textit{Contr.}$$

$$\frac{\Gamma, x\colon A \rhd M\colon B}{\Gamma, y\colon !A \rhd M[x := \mathsf{derelict}(y)]\colon B}\ \textit{Dereliction}$$

$$\frac{\vec{x}\colon !\Gamma \rhd M\colon A}{\vec{y}\colon !\Gamma \rhd \mathsf{promote}\ \vec{y}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M\colon !A}\ \textit{Promotion}$$

Figure 3.2: Term Assignment for Sequent Calculus Formulation of **ILL**

$$\frac{\vec{x}: !\Gamma \triangleright M: A}{\vec{y}: !\Gamma \triangleright \mathsf{promote}\ \vec{y}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M: !A}\ \mathit{Promotion.}$$

We give the term assignment for the entire sequent calculus formulation in Figure 3.2

As we might expect the translation procedures, $\mathcal{N}$ and $\mathcal{S}$, relating the natural deduction and sequent calculus formulations of **ILL** extend to the term assignment systems.

Theorem 14.

- If $\vdash^S \Gamma \triangleright \overset{\pi}{M}: A$ then $\vdash^N \Gamma \triangleright \overset{\mathcal{N}(\pi)}{M}: A$.
- If $\vdash^N \Gamma \triangleright \overset{\mathcal{D}}{M}: A$ then $\vdash^S \Gamma \triangleright \overset{\mathcal{S}(\mathcal{D})}{M}: A$.

Proof. By straightforward structural induction ■

Thus we have that the term assignment systems are equivalent.

Corollary 2. $\vdash^N\ \Gamma \triangleright M: A$ iff $\vdash^S\ \Gamma \triangleright M: A$.

Although as we stated above there is no real Curry-Howard correspondence for the sequent calculus, we can actually define a syntax, which we shall call a *Sequent Term Assignment*, which identifies every different sequent proof. Essentially this means that every proof rule introduces a new piece of syntax, including the *Cut* rule.[2] Although probably not directly useful as a term calculus for programming, a sequent term assignment is of use to those interested in automated deduction, where the actual sequent calculus *derivation* is of primary importance and thus the term encodes precisely this derivation. We give the sequent term assignment system in Figure 3.3.

Terms from a sequent term assignment system have the following useful property.

Proposition 4. If a term M from the sequent term assignment contains no occurrence of the cut term constructor then it denotes a cut-free proof.

Hence we can see whether a proof is cut-free by simply examining the term. It also means that all possible (primary) redexes are explicit within the term. The main problem with such a sequent term assignment system is the number of reduction rules. Every step in the cut-elimination process entails a reduction at the term level: in the case of **ILL** this would mean 361 reduction rules!

3 Linear Combinatory Logic

We can apply a Curry-Howard correspondence to the axiomatic formulation given in Section 3 of Chapter 2 to obtain a combinatory logic. Thus derivations are of the form $\vec{x}: \Gamma \Rightarrow e: A$, where variables and axioms are named and the derivation has an associated combinatory term, e. We give the combinatory formulation of **ILL** in Figure 3.4.

As noted in §3 of Chapter 2, the *Promotion* rule has the side condition that the context must be *empty* when it is applied. Given that in combinatory logic we have variable names explicit we might have imagined a rule such as that from the natural deduction formulation, for example,

$$\frac{\vec{x_1} \Rightarrow e_1: !A_1\ \ \dots\ \ \vec{x_n} \Rightarrow e_n: !A_n \qquad y_1: !A_1, \dots, y_n: !A_n \Rightarrow f: B}{\vec{x_1}: \Gamma_1, \dots, \vec{x_n}: \Gamma_n \Rightarrow \mathtt{prom}\ \vec{e}\ \mathtt{for}\ \vec{y}\ \mathtt{in}\ f: !B}\ \mathit{Promotion'}.$$

However, the practical concerns alluded to earlier stem from the fact that reduction in a combinatory formulation involves *no substitution of terms for variables*. Clearly the formulation given above would not have that property and for that reason we shall not consider it further.

[2] Again we shall not introduce explicit syntax for the *Exchange* rule in favour of adopting the convention that the contexts are multisets.

$$\frac{}{x\colon A \rhd x\colon A}\ \textit{Identity}$$

$$\frac{\Gamma \rhd M\colon B \qquad x\colon B, \Delta \rhd N\colon C}{\Gamma, \Delta \rhd \mathsf{cut}\ M\ \mathsf{for}\ x\ \mathsf{in}\ N\colon C}\ \textit{Cut}$$

$$\frac{}{\vec{x}\colon \Gamma, y\colon \mathbf{f} \rhd \mathsf{false}_A(\vec{x}\ ;\ y)\colon A}\ (\mathbf{f}_{\mathcal{L}}) \qquad \frac{}{\vec{x}\colon \Gamma \rhd \mathsf{true}(\vec{x})\colon \mathbf{t}}\ (\mathbf{t}_{\mathcal{R}})$$

$$\frac{\Gamma \rhd M\colon A}{\Gamma, x\colon I \rhd \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ M\colon A}\ (I_{\mathcal{L}}) \qquad \frac{}{\rhd *\colon I}\ (I_{\mathcal{R}})$$

$$\frac{\Gamma, x\colon A, y\colon B \rhd M\colon C}{\Gamma, z\colon A{\otimes}B \rhd \mathsf{let}\ z\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ M\colon C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon B}{\Gamma, \Delta \rhd M{\otimes}N\colon A{\otimes}B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \rhd M\colon A \qquad \Delta, x\colon B \rhd N\colon C}{\Gamma, \Delta, y\colon A{\multimap}B \rhd \mathsf{let}\ y\ \mathsf{be}\ M{\multimap}x\ \mathsf{in}\ N\colon C}\ ({\multimap}_{\mathcal{L}}) \qquad \frac{\Gamma, x\colon A \rhd M\colon B}{\Gamma \rhd \lambda x\colon A.M\colon A{\multimap}B}\ ({\multimap}_{\mathcal{R}})$$

$$\frac{\Gamma, x\colon A \rhd M\colon C}{\Gamma, y\colon A\&B \rhd \mathsf{let}\ y\ \mathsf{be}\ x\&\_\ \mathsf{in}\ M\colon C}\ (\&_{\mathcal{L}-1})$$

$$\frac{\Gamma, x\colon B \rhd M\colon C}{\Gamma, y\colon A\&B \rhd \mathsf{let}\ y\ \mathsf{be}\ \_\&x\ \mathsf{in}\ M\colon C}\ (\&_{\mathcal{L}-2})$$

$$\frac{\vec{x}\colon \Gamma \rhd M\colon A \qquad \vec{x}\colon \Gamma \rhd N\colon B}{\vec{x}\colon \Gamma \rhd \langle M, N\rangle\colon A\&B}\ (\&_{\mathcal{R}})$$

$$\frac{\vec{x}\colon \Gamma, y\colon A \rhd M\colon C \qquad \vec{x}\colon \Gamma, z\colon B \rhd N\colon C}{\vec{x}\colon \Gamma, w\colon A \oplus B \rhd \mathsf{case}\ w\ \mathsf{of}\ \mathsf{inl}(y)\ \rightarrow\ M\ \|\ \mathsf{inr}(z)\ \rightarrow\ N\colon C}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Gamma \rhd M\colon A}{\Gamma \rhd \mathsf{inl}(M)\colon A \oplus B}\ (\oplus_{\mathcal{R}-1}) \qquad \frac{\Gamma \rhd M\colon B}{\Gamma \rhd \mathsf{inr}(M)\colon A \oplus B}\ (\oplus_{\mathcal{R}-2})$$

$$\frac{\Gamma \rhd M\colon B}{\Gamma, x\colon !A \rhd \mathsf{discard}\ x\ \mathsf{in}\ M\colon B}\ \textit{Weakening} \qquad \frac{\Gamma, y\colon !A, z\colon !A \rhd M\colon B}{\Gamma, x\colon !A \rhd \mathsf{copy}\ x\ \mathsf{as}\ y, z\ \mathsf{in}\ M\colon B}\ \textit{Contr.}$$

$$\frac{\Gamma, x\colon A \rhd M\colon B}{\Gamma, y\colon !A \rhd \mathsf{derelict}\ y\ \mathsf{for}\ x\ \mathsf{in}\ M\colon B}\ \textit{Dereliction} \qquad \frac{!\Gamma \rhd M\colon A}{!\Gamma \rhd \mathsf{promote}(M)\colon !A}\ \textit{Promotion}$$

Figure 3.3: Sequent Term Assignment for **ILL**

$$
\begin{array}{rcl}
\mathtt{I}_A & : & A \multimap A \\
\mathtt{B}_{A,B,C} & : & (B \multimap C) \multimap ((A \multimap B) \multimap (A \multimap C)) \\
\mathtt{C}_{A,B,C} & : & (A \multimap (B \multimap C)) \multimap (B \multimap (A \multimap C)) \\
\\
\mathtt{tensor}_{A,B} & : & A \multimap (B \multimap A \otimes B) \\
\mathtt{split}_{A,B,C} & : & A \otimes B \multimap ((A \multimap (B \multimap C)) \multimap C) \\
\\
\mathtt{unit} & : & I \\
\mathtt{let}_A & : & A \multimap (I \multimap A) \\
\\
\mathtt{fst}_{A,B} & : & A \& B \multimap A \\
\mathtt{snd}_{A,B} & : & A \& B \multimap B \\
\mathtt{wdist}_{A,B,C} & : & (A \multimap B) \& (A \multimap C) \multimap (A \multimap B \& C) \\
\\
\mathtt{inl}_{A,B} & : & A \multimap A \oplus B \\
\mathtt{inr}_{A,B} & : & B \multimap A \oplus B \\
\mathtt{sdist}_{A,B,C} & : & (A \multimap C) \& (B \multimap C) \multimap (A \oplus B \multimap C) \\
\\
\mathtt{init}_A & : & \mathbf{f} \multimap A \\
\mathtt{term}_A & : & A \multimap \mathbf{t} \\
\\
\mathtt{dupl}_{A,B} & : & (!A \multimap (!A \multimap B)) \multimap (!A \multimap B) \\
\mathtt{disc}_{A,B} & : & B \multimap (!A \multimap B) \\
\mathtt{eps}_A & : & !A \multimap A \\
\mathtt{delta}_A & : & !A \multimap !!A \\
\mathtt{edist}_{A,B} & : & !(A \multimap B) \multimap (!A \multimap !B)
\end{array}
$$

$$\frac{}{x\colon A \Rightarrow x\colon A}\ \mathit{Identity} \qquad \frac{}{\Rightarrow \mathtt{c}\colon A}\ \mathit{Axiom}\ \text{(where c is taken from above)}$$

$$\frac{\Gamma \Rightarrow e\colon A \multimap B \quad \Delta \Rightarrow f\colon A}{\Gamma, \Delta \Rightarrow ef\colon B}\ \mathit{Modus\ Ponens} \qquad \frac{\vec{x}\colon \Gamma \Rightarrow e\colon A \quad \vec{x}\colon \Gamma \Rightarrow f\colon B}{\vec{x}\colon \Gamma \Rightarrow \mathtt{with}(e,f)\colon A \& B}\ \mathit{With}$$

$$\frac{\Rightarrow e\colon A}{\Rightarrow \mathtt{promote}(e)\colon !A}\ \mathit{Promotion}$$

Figure 3.4: Combinatory Formulation of **ILL**

(It should also be noted that in the context of this thesis we are considering a *typed* combinatory logic. Thus the axioms are explicitly typed, although in places we shall omit the type information in favour of brevity. For example, where we should write $\texttt{delta}_A\colon !A \multimap !!A$ we have written $\texttt{delta}\colon !A \multimap !!A$.)
The proof of Theorem 6 in Chapter 2, when viewed at the level of combinatory logic actually gives a process of 'abstracting' a variable from a combinatory term. Let us repeat the theorem at the level of terms.

Theorem 15. If $\Gamma, x{:}A \Rightarrow e{:}B$ then there exists a combinatory term, $[x]e$,[3] such that $\Gamma \Rightarrow [x]e{:}A \multimap B$.

The proof of this theorem yields the process given in Figure 3.5

$$\begin{array}{rcll}
[x]x & \stackrel{\text{def}}{=} & \texttt{I} & \\
[x]ef & \stackrel{\text{def}}{=} & \texttt{C}\,([x]e)\,f & \text{If } x \in FV(e) \\
[x]ef & \stackrel{\text{def}}{=} & \texttt{B}\,e\,([x]f) & \text{Otherwise} \\
[x](\texttt{with}(e,f)) & \stackrel{\text{def}}{=} & \texttt{wdist}(\texttt{with}([x]e,[x]f)) &
\end{array}$$

where FV is defined:

$$\begin{array}{rcll}
FV(x) & \stackrel{\text{def}}{=} & \{x\} & \\
FV(ef) & \stackrel{\text{def}}{=} & FV(e) \cup FV(f) & \\
FV(\texttt{with}(e,f)) & \stackrel{\text{def}}{=} & FV(e) & (= FV(f)) \\
FV(\texttt{promote}(e)) & \stackrel{\text{def}}{=} & \emptyset &
\end{array}$$

Figure 3.5: Variable Abstraction

4 Reduction Rules

In Chapter 2 we presented two processes for normalizing proofs: β-reduction for natural deduction and cut elimination for sequent calculus. In addition we found a need for some extra reduction steps, the commuting conversions, for the natural deduction formulation. In this section we shall review each of these processes in turn and present them at the level of terms.

4.1 Normalization

As discussed in §2.1 in Chapter 2, the β-reduction rules arise when a connective is introduced only to be immediately eliminated. We can apply the Curry-Howard correspondence to these reduction rules to produce term reduction rules, which are given in Figure 3.6.[4]

An immediate property of this relation is known as *subject reduction*, which basically states that well-typed terms reduce to well-typed terms. Both O'Hearn [59] and Lincoln and Mitchell [55] have shown that this property does not hold for previous proposals for the linear term calculus.

Theorem 16. If $\Gamma \triangleright M{:}A$ and $M \leadsto_\beta N$ then $\Gamma \triangleright N{:}A$.

Proof. By induction on $\Gamma \triangleright M{:}A$ and use of Lemma 3. ■

[3] $[x]e$ is written as $\lambda^* x.e$ by Barendregt [8].

[4] We shall use the standard notation of $\leadsto_\beta^+$ to represent the transitive closure of $\leadsto_\beta$, and $\leadsto_\beta^*$ to represent the reflexive, transitive closure of $\leadsto_\beta$.

$$
\begin{array}{rcl}
(\lambda x{:}\,A.M)\;N & \rightsquigarrow_\beta & M[x := N] \\
\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ M & \rightsquigarrow_\beta & M \\
\mathsf{let}\ M\otimes N\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ P & \rightsquigarrow_\beta & P[x := M, y := N] \\
\mathsf{fst}(\langle M, N\rangle) & \rightsquigarrow_\beta & M \\
\mathsf{snd}(\langle M, N\rangle) & \rightsquigarrow_\beta & N \\
\mathsf{case}\ (\mathsf{inl}(M))\ \mathsf{of} & & \\
\mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P & \rightsquigarrow_\beta & N[x := M] \\
\mathsf{case}\ (\mathsf{inr}(M))\ \mathsf{of} & & \\
\mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P & \rightsquigarrow_\beta & P[y := M] \\
\mathsf{derelict}(\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N) & \rightsquigarrow_\beta & N[\vec{x} := \vec{M}] \\
\mathsf{discard}\ (\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N)\ \mathsf{in}\ P & \rightsquigarrow_\beta & \mathsf{discard}\ \vec{M}\ \mathsf{in}\ P \\
\mathsf{copy}\ (\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N)\ \mathsf{as}\ y, z\ \mathsf{in}\ P & \rightsquigarrow_\beta & \mathsf{copy}\ \vec{M}\ \mathsf{as}\ \vec{u}, \vec{v}\ \mathsf{in} \\
 & & \quad P[\ y := \mathsf{promote}\ \vec{u}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N, \\
 & & \qquad z := \mathsf{promote}\ \vec{v}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N]
\end{array}
$$

Figure 3.6: β-reduction rules

4.2 Commuting Conversions

In §2.2 of Chapter 2 we showed how a consideration of the subformula property induces a further set of reductions which we term commuting conversions. Again we can apply the Curry-Howard correspondence to these conversions to produce a set of reduction rules at the level of terms, which we give in Figures 3.7, 3.8, 3.9 and 3.10.

4.3 Cut Elimination

In Chapter 2 we gave a detailed proof of cut elimination for the sequent calculus formulation. This essentially consisted of considering every possibility of a (R_1, R_2)-cut and showing how it could be replaced by 'simpler' cut(s). At a term level, we have a (R_1, R_2)-cut of the form

$$
\dfrac{\dfrac{}{\Delta \triangleright M{:}\,A}\,R_1 \qquad \dfrac{}{\Gamma, x{:}\,A \triangleright N{:}\,B}\,R_2}{\Gamma, \Delta \triangleright N[x := M]{:}\,B}\,\mathit{Cut}
$$

which we replace with a proof ending with

$$\Gamma, \Delta \triangleright P{:}\,B$$

We shall write $N[x := M] \rightsquigarrow_{cut} P$ to represent this term reduction. For **ILL** there are 19 rules besides *Cut* (and *Exchange*) and hence 361 possibilities for (R_1, R_2)-cuts. Fortunately most of these possibilities are not computationally significant in the sense that they have no effect on the terms. We shall use the following definition.

Definition 16. A cut-reduction $N[x := M] \rightsquigarrow_{cut} P$ is *insignificant* if $N[x := M] =_\alpha P$.

Thus we shall classify cut-reductions by whether they are significant or not. We shall consider each sort of Cut in turn, identifying at first those which they are significant or not.

- Any cut involving the *Identity* rule is insignificant. There are 37 of these.
- A $(X_\mathcal{L}, Y_\mathcal{L})$-cut. There are 100 of these. The insignificant cuts are of the form $(\multimap_\mathcal{L}, Y_\mathcal{L})$, $(\&_{\mathcal{L}-1}, Y_\mathcal{L})$, $(\&_{\mathcal{L}-2}, Y_\mathcal{L})$, $(\mathit{Dereliction}, Y_\mathcal{L})$, where $Y \neq \mathbf{f}$, which amounts to 36 cut reductions. Thus there are 64 significant remaining cut-reductions.

$$
\begin{array}{rcl}
(\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N)P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (NP) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N)\ \mathsf{be}\ *\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N)\ \mathsf{be}\ w{\otimes}z\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ w{\otimes}z\ \mathsf{in}\ P) \\
\mathsf{fst}(\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ \mathsf{fst}(N) \\
\mathsf{snd}(\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ \mathsf{snd}(N) \\
\mathsf{false}_D(\vec{P}\,;\ \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (\mathsf{false}_D(\vec{P}\,;\ N)) \\
\mathsf{case}\ (\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N)\ \mathsf{of} & & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in} \\
\mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q & \rightsquigarrow_c & (\mathsf{case}\ N\ \mathsf{of}\ \mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q) \\
\mathsf{discard}\ (\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N)\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (\mathsf{discard}\ N\ \mathsf{in}\ P) \\
\mathsf{copy}\ (\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N)\ \mathsf{as}\ w,z\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (\mathsf{copy}\ N\ \mathsf{as}\ w,z\ \mathsf{in}\ P) \\
\mathsf{derelict}(\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (\mathsf{derelict}(N)) \\
\\
(\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N)P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ (NP) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N)\ \mathsf{be}\ *\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N)\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ P) \\
\mathsf{fst}(\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ \mathsf{fst}(N) \\
\mathsf{snd}(\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ \mathsf{snd}(N) \\
\mathsf{false}_D(\vec{P}\,;\ \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{false}_D(\vec{P}\,;\ N)) \\
\mathsf{case}\ (\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N)\ \mathsf{of} & & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in} \\
\mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q & \rightsquigarrow_c & (\mathsf{case}\ N\ \mathsf{of}\ \mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q) \\
\mathsf{discard}\ (\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N)\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{discard}\ N\ \mathsf{in}\ P) \\
\mathsf{copy}\ (\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N)\ \mathsf{as}\ x,y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{copy}\ N\ \mathsf{as}\ x,y\ \mathsf{in}\ P) \\
\mathsf{derelict}(\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{derelict}(N)) \\
\\
(\mathsf{discard}\ M\ \mathsf{in}\ N)P & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ (NP) \\
\mathsf{let}\ (\mathsf{discard}\ M\ \mathsf{in}\ N)\ \mathsf{be}\ *\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{discard}\ M\ \mathsf{in}\ N)\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ P) \\
\mathsf{fst}(\mathsf{discard}\ M\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ \mathsf{fst}(N) \\
\mathsf{snd}(\mathsf{discard}\ M\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ \mathsf{snd}(N) \\
\mathsf{false}_C(\vec{P}\,;\ \mathsf{discard}\ M\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ (\mathsf{false}_C(\vec{P}\,;\ N)) \\
\mathsf{case}\ (\mathsf{discard}\ M\ \mathsf{in}\ N)\ \mathsf{of} & & \mathsf{discard}\ M\ \mathsf{in} \\
\mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q & \rightsquigarrow_c & (\mathsf{case}\ N\ \mathsf{of}\ \mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q) \\
\mathsf{discard}\ (\mathsf{discard}\ M\ \mathsf{in}\ N)\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ (\mathsf{discard}\ N\ \mathsf{in}\ P) \\
\mathsf{copy}\ (\mathsf{discard}\ M\ \mathsf{in}\ N)\ \mathsf{as}\ x,y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ (\mathsf{copy}\ N\ \mathsf{as}\ x,y\ \mathsf{in}\ P) \\
\mathsf{derelict}(\mathsf{discard}\ M\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{discard}\ M\ \mathsf{in}\ (\mathsf{derelict}(N))
\end{array}
$$

Figure 3.7: Commuting Conversions I

$$
\begin{array}{rcl}
(\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N)P & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ (NP) \\
\mathsf{let}\ (\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N)\ \mathsf{be}\ *\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ P) \\
\mathsf{let}\ (\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N)\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ (\mathsf{let}\ N\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ P) \\
\mathsf{fst}(\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ \mathsf{fst}(N) \\
\mathsf{snd}(\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ \mathsf{snd}(N) \\
\mathsf{false}_C(\vec{P}\,;\ \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ (\mathsf{false}_C(\vec{P}\,;\ N)) \\
\begin{array}{c}\mathsf{case}\ (\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N)\ \mathsf{of} \\ \mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q\end{array} & \rightsquigarrow_c & \begin{array}{c}\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in} \\ (\mathsf{case}\ N\ \mathsf{of}\ \mathsf{inl}(w) \to P \parallel \mathsf{inr}(z) \to Q)\end{array} \\
\mathsf{discard}\ (\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N)\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ (\mathsf{discard}\ N\ \mathsf{in}\ P) \\
\mathsf{copy}\ (\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N)\ \mathsf{as}\ x,y\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ (\mathsf{copy}\ N\ \mathsf{as}\ x,y\ \mathsf{in}\ P) \\
\mathsf{derelict}(\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N) & \rightsquigarrow_c & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ (\mathsf{derelict}(N))
\end{array}
$$

$$
\begin{array}{rcll}
\mathsf{let}\ (\mathsf{false}_I(\vec{P}\,;\ N))\ \mathsf{be}\ *\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{false}_A(\vec{P},\vec{x}\,;\ N) & (\vec{x} = FV(P)) \\
(\mathsf{false}_{A\multimap B}(\vec{P}\,;\ N))P & \rightsquigarrow_c & \mathsf{false}_B(\vec{P},\vec{x}\,;\ N) & (\vec{x} = FV(P)) \\
\mathsf{let}\ (\mathsf{false}_{A\otimes B}(\vec{P}\,;\ N))\ \mathsf{be}\ y\otimes z\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{false}_C(\vec{P},\vec{x}\,;\ N) & (\vec{x} = FV(P) - \{y,z\}) \\
\mathsf{fst}(\mathsf{false}_{A\&B}(\vec{P}\,;\ N)) & \rightsquigarrow_c & \mathsf{false}_A(\vec{P}\,;\ N) & \\
\mathsf{snd}(\mathsf{false}_{A\&B}(\vec{P}\,;\ N)) & \rightsquigarrow_c & \mathsf{false}_B(\vec{P}\,;\ N) & \\
\mathsf{false}_A(\vec{P}\,;\ \mathsf{false}_{\mathbf{f}}(\vec{Q}\,;\ N)) & \rightsquigarrow_c & \mathsf{false}_A(\vec{P},\vec{y}\,;\ N) & \\
\begin{array}{c}\mathsf{case}\ (\mathsf{false}_{A\oplus B}(\vec{P}\,;\ N))\ \mathsf{of} \\ \mathsf{inl}(x) \to P \parallel \mathsf{inr}(y) \to Q\end{array} & \rightsquigarrow_c & \mathsf{false}_C(\vec{P},\vec{w}\,;\ N) & (\vec{w} = FV(P) - \{x\}) \\
\mathsf{discard}\ (\mathsf{false}_{!A}(\vec{P}\,;\ N))\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{false}_B(\vec{P},\vec{x}\,;\ N) & (\vec{x} = FV(P)) \\
\mathsf{copy}\ (\mathsf{false}_{!A}(\vec{P}\,;\ N))\ \mathsf{as}\ y,z\ \mathsf{in}\ P & \rightsquigarrow_c & \mathsf{false}_B(\vec{P},\vec{x}\,;\ N) & (\vec{x} = FV(P) - \{y,z\}) \\
\mathsf{derelict}(\mathsf{false}_{!A}(\vec{P}\,;\ N)) & \rightsquigarrow_c & \mathsf{false}_A(\vec{P}\,;\ N) &
\end{array}
$$

Figure 3.8: Commuting Conversions II

$$\begin{array}{lcl}
(\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P)Q & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to (NQ) \parallel \mathsf{inl}(y) \to (PQ)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{let}\ (\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P)\ \mathsf{be}\ *\ \mathsf{in}\ Q & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to (\mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ Q)\ \ \mathsf{inr}(y) \to (\mathsf{let}\ P\ \mathsf{be}\ *\ \mathsf{in}\ Q)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{let}\ (\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P)\ \mathsf{be}\ w \otimes z\ \mathsf{in}\ Q & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to (\mathsf{let}\ N\ \mathsf{be}\ w \otimes z\ \mathsf{in}\ Q)\ \ \mathsf{inr}(y) \to (\mathsf{let}\ P\ \mathsf{be}\ w \otimes z\ \mathsf{in}\ Q)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{fst}(\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P) & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to \mathsf{fst}(N)\ \ \mathsf{inr}(y) \to \mathsf{fst}(P)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{snd}(\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P) & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to \mathsf{snd}(N)\ \ \mathsf{inr}(y) \to \mathsf{snd}(P)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{false}_C(\vec{P};\ (\mathsf{case}\ N\ \mathsf{of}\ \mathsf{inl}(x) \to P \parallel \mathsf{inr}(y) \to Q)) & \rightsquigarrow_c & \mathsf{case}\ N\ \mathsf{of}\ \mathsf{inl}(x) \to (\mathsf{false}_C(\vec{P}\,;\ P))\ \ \mathsf{inr}(y) \to (\mathsf{false}_C(\vec{P}\,;\ Q))
\end{array}$$

$$\begin{array}{lcl}
\mathsf{case}\ (\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P)\ \mathsf{as}\ \mathsf{inl}(w) \to Q \parallel \mathsf{inr}(z) \to R & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{as}\ \mathsf{inl}(x) \to (\mathsf{case}\ N\ \mathsf{of}\ \mathsf{inl}(w) \to Q\ \ \mathsf{inr}(z) \to R)\ \ \mathsf{inr}(y) \to (\mathsf{case}\ P\ \mathsf{of}\ \mathsf{inl}(w) \to Q\ \ \mathsf{inr}(z) \to R)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{discard}\ (\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P)\ \mathsf{in}\ Q & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to (\mathsf{discard}\ N\ \mathsf{in}\ Q)\ \ \mathsf{inr}(y) \to (\mathsf{discard}\ P\ \mathsf{in}\ Q)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{copy}\ (\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P)\ \mathsf{as}\ w, z\ \mathsf{in}\ Q & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to (\mathsf{copy}\ N\ \mathsf{as}\ w, z\ \mathsf{in}\ Q)\ \ \mathsf{inr}(y) \to (\mathsf{copy}\ P\ \mathsf{as}\ w, z\ \mathsf{in}\ Q)
\end{array}$$

$$\begin{array}{lcl}
\mathsf{derelict}(\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P) & \rightsquigarrow_c & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to (\mathsf{derelict}(N))\ \ \mathsf{inr}(y) \to (\mathsf{derelict}(P))
\end{array}$$

Figure 3.9: Commuting Conversions III

$$\begin{array}{l}
\mathsf{promote}\ (\mathsf{promote}\ \vec{x'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M), y_2', \ldots, y_n'\ \mathsf{for}\ y_1, \ldots, y_n\ \mathsf{in}\ N \\
\leadsto_c\ \mathsf{promote}\ \vec{x'}, y_2', \ldots, y_n'\ \mathsf{for}\ \vec{x''}, y_2, \ldots, y_n\ \mathsf{in}\ N[y_1 := \mathsf{promote}\ \vec{x''}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M]
\end{array}$$

$$\begin{array}{l}
\mathsf{promote}\ \vec{w}, (\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N \\
\leadsto_c\ \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ (\mathsf{promote}\ \vec{w}, M\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N)
\end{array}$$

$$\begin{array}{l}
\mathsf{promote}\ \vec{z}, (\mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N \\
\leadsto_c\ \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{promote}\ \vec{z}, M\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N)
\end{array}$$

$$\begin{array}{l}
\mathsf{promote}\ \vec{w}, (\mathsf{case}\ z\ \mathsf{of}\ \mathsf{inl}(x)\ \rightarrow\ M \parallel \mathsf{inr}(y)\ \rightarrow\ N)\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ P \\
\leadsto_c\ \mathsf{case}\ z\ \mathsf{of} \\
\qquad\qquad \mathsf{inl}(x) \rightarrow \mathsf{promote}\ \vec{w}, M\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ P \\
\qquad\qquad \mathsf{inr}(y) \rightarrow \mathsf{promote}\ \vec{w}, N\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ P
\end{array}$$

$$\begin{array}{l}
\mathsf{promote}\ \vec{z}, (\mathsf{discard}\ x\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N \\
\leadsto_c\ \mathsf{discard}\ x\ \mathsf{in}\ (\mathsf{promote}\ \vec{z}, M\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N)
\end{array}$$

$$\begin{array}{l}
\mathsf{promote}\ \vec{w}, (\mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N \\
\leadsto_c\ \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ (\mathsf{promote}\ \vec{w}, M\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N)
\end{array}$$

$$\left.\begin{array}{l}
\mathsf{true}(M, \vec{x}) \\
\leadsto_c\ \mathsf{true}(\vec{y}, \vec{x}) \\
\\
\mathsf{false}_A(M\ ;\ x) \\
\leadsto_c\ \mathsf{false}_A(\vec{y}\ ;\ x)
\end{array}\right\}\ \text{where } FV(M) = \vec{y}.$$

Figure 3.10: Commuting Conversions IV

- A $(X_{\mathcal{R}}, Y_{\mathcal{R}})$-cut. There are 64 of these. They are all insignificant except for the case of (*Promotion*, *Promotion*) and $(X_{\mathcal{R}}, \mathbf{t}_{\mathcal{R}})$.
- A $(X_{\mathcal{L}}, Y_{\mathcal{R}})$-cut. There are 80 of these. There are 15 significant cut reductions which those of the form $(\otimes_{\mathcal{L}}, \mathit{Promotion})$, $(I_{\mathcal{L}}, \mathit{Promotion})$, $(\oplus_{\mathcal{L}}, \mathit{Promotion})$, (*Weakening*, *Promotion*), (*Contraction*, *Promotion*) and $(X_{\mathcal{L}}, \mathbf{t}_{\mathcal{R}})$. The remaining 65 cut-reductions are insignificant.
- A $(X_{\mathcal{R}}, Y_{\mathcal{L}})$-cut. There are 80 of these. Cuts of the form $(X_{\mathcal{R}}, X_{\mathcal{L}})$ and $(X_{\mathcal{R}}, \mathbf{f}_{\mathcal{L}})$, of which there are 18, are significant. The remaining 62 cut-reductions are insignificant.

We shall classify these significant cut-reductions and then consider them in turn.

Principal Cuts. These are cut-reductions of the form $(X_{\mathcal{R}}, X_{\mathcal{L}})$.

Secondary Cuts.
These are cut-reductions of the following form: (*Promotion*, *Promotion*), $(\otimes_{\mathcal{L}}, \mathit{Promotion})$, $(I_{\mathcal{L}}, \mathit{Promotion})$, $(\oplus_{\mathcal{L}}, \mathit{Promotion})$, (*Weakening*, *Promotion*) and (*Contraction*, *Promotion*).

Commuting Cuts. These are cuts of the form $(X_{\mathcal{L}}, Y_{\mathcal{L}})$, except where X is either $\multimap$, &, $\mathbf{f}$ or *Dereliction* or where Y is $\mathbf{f}$.

Unit Cuts. These are cuts of the form $(X, \mathbf{f}_{\mathcal{L}})$ and $(X, \mathbf{t}_{\mathcal{R}})$.

Principal Cuts

There are 10 of these and we shall consider them in turn.

- $(\otimes_{\mathcal{R}}, \otimes_{\mathcal{L}})$-cut.

$$\dfrac{\dfrac{\Delta \vdash A \qquad \Theta \vdash B}{\Delta, \Theta \vdash A \otimes B}(\otimes_{\mathcal{R}}) \qquad \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}(\otimes_{\mathcal{L}})}{\Gamma, \Delta, \Theta \vdash C}\mathit{Cut}$$

This reduces to

$$\dfrac{\Theta \vdash B \qquad \dfrac{\Delta \vdash A \qquad \Gamma, A, B \vdash C}{\Gamma, \Delta, B \vdash C}\mathit{Cut}}{\Gamma, \Delta, \Theta \vdash C}\mathit{Cut.}$$

At the level of terms this gives the cut-reduction

$$\mathsf{let}\ M \otimes N\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ P \quad \rightsquigarrow_{cut} \quad P[x := M, y := N].$$

- $(I_{\mathcal{R}}, I_{\mathcal{L}})$-cut.

$$\dfrac{\dfrac{}{\vdash I}(I_{\mathcal{R}}) \qquad \dfrac{\Gamma \vdash A}{\Gamma, I \vdash A}(I_{\mathcal{L}})}{\Gamma \vdash A}\mathit{Cut}$$

This reduces to

$$\Gamma \vdash A.$$

At the level of terms this gives the cut-reduction

$$\mathsf{let}\ * \ \mathsf{be}\ * \ \mathsf{in}\ M \quad \rightsquigarrow_{cut} \quad M.$$

- $(\multimap_{\mathcal{R}}, \multimap_{\mathcal{L}})$-cut.

$$\dfrac{\dfrac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\,(\multimap_{\mathcal{R}}) \qquad \dfrac{\Gamma \vdash A \qquad B, \Theta \vdash C}{\Gamma, A \multimap B, \Theta \vdash C}\,(\multimap_{\mathcal{L}})}{\Gamma, \Delta, \Theta \vdash C}\,\mathit{Cut}$$

This reduces to

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Delta, A \vdash B \qquad B, \Theta \vdash C}{\Delta, A, \Theta \vdash C}\,\mathit{Cut}}{\Gamma, \Delta, \Theta \vdash C}\,\mathit{Cut}.$$

At the level of terms this gives the cut-reduction

$$P[y := ((\lambda x{:}\, A.M)N)] \quad \leadsto_{cut} \quad P[y := (M[x := N])].$$

- $(\oplus_{\mathcal{R}-1}, \oplus_L)$-cut.

$$\dfrac{\dfrac{\Delta \vdash A}{\Delta \vdash A \oplus B}\,(\oplus_{\mathcal{R}-1}) \qquad \dfrac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,(\oplus_{\mathcal{R}})}{\Gamma, \Delta \vdash C}\,\mathit{Cut}$$

This reduces to

$$\dfrac{\Delta \vdash A \qquad \Gamma, A \vdash C}{\Gamma, \Delta \vdash C}\,\mathit{Cut}.$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{case\ inl}(M)\ \mathsf{of\ inl}(x) \ \to\ N \,\|\, \mathsf{inr}(y) \ \to\ P \\ \leadsto_{cut}\ N[x := M]. \end{array}$$

- $(\oplus_{\mathcal{R}-2}, \oplus_L)$-cut.

$$\dfrac{\dfrac{\Delta \vdash B}{\Delta \vdash A \oplus B}\,(\oplus_{\mathcal{R}-2}) \qquad \dfrac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,(\oplus_{\mathcal{R}})}{\Gamma, \Delta \vdash C}\,\mathit{Cut}$$

This reduces to

$$\dfrac{\Delta \vdash B \qquad \Gamma, B \vdash C}{\Gamma, \Delta \vdash C}\,\mathit{Cut}.$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{case\ inr}(M)\ \mathsf{of\ inl}(x) \ \to\ N \,\|\, \mathsf{inr}(y) \ \to\ P \\ \leadsto_{cut}\ P[y := M]. \end{array}$$

- $(\&_{\mathcal{R}}, \&_{\mathcal{L}-1})$-cut.

$$\dfrac{\dfrac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A\&B}(\&_{\mathcal{R}}) \qquad \dfrac{\Gamma, A \vdash C}{\Gamma, A\&B \vdash C}(\&_{\mathcal{L}-1})}{\Gamma, \Delta \vdash C}\,Cut$$

This reduces to

$$\dfrac{\Delta \vdash A \qquad \Gamma, A \vdash C}{\Gamma, \Delta \vdash C}\,Cut.$$

At the level of terms this amounts to the cut-reduction

$$P[x := \mathsf{fst}(\langle M, N\rangle)] \quad \leadsto_{cut} \quad P[x := M].$$

- $(\&_{\mathcal{R}}, \&_{\mathcal{L}-2})$-cut.

$$\dfrac{\dfrac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A\&B}(\&_{\mathcal{R}}) \qquad \dfrac{\Gamma, B \vdash C}{\Gamma, A\&B \vdash C}(\&_{\mathcal{L}-2})}{\Gamma, \Delta \vdash C}\,Cut.$$

This reduces to

$$\dfrac{\Delta \vdash B \qquad \Gamma, B \vdash C}{\Gamma, \Delta \vdash C}\,Cut.$$

At the level of terms this amounts to the cut-reduction

$$P[x := \mathsf{snd}(\langle M, N\rangle)] \quad \leadsto_{cut} \quad P[x := N].$$

- $(Promotion, Dereliction)$-cut.

$$\dfrac{\dfrac{!\Delta \vdash A}{!\Delta \vdash !A}\,Promotion \qquad \dfrac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,Dereliction}{\Gamma, !\Delta \vdash B}\,Cut$$

This reduces to

$$\dfrac{!\Delta \vdash A \qquad \Gamma, A \vdash B}{\Gamma, !\Delta \vdash B}\,Cut.$$

At the level of terms this gives the cut-reduction

$$N[z := \mathsf{derelict}(\mathsf{promote}\,\vec{y}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,M)] \quad \leadsto_{cut} \quad N[z := M[\vec{x} := \vec{y}]].$$

- (*Promotion*, *Weakening*)-cut.

$$\dfrac{\dfrac{!\Delta \vdash A}{!\Delta \vdash !A}\,Promotion \qquad \dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}\,Weakening}{\Gamma, !\Delta \vdash B}\,Cut$$

This reduces to

$$\dfrac{\Gamma \vdash B}{\Gamma, !\Delta \vdash B}\,Weakening^{*}.$$

At the level of terms this gives the following cut-reduction

$$\mathsf{discard}\ (\mathsf{promote}\ \vec{y}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M)\ \mathsf{in}\ N \quad \leadsto_{cut} \quad \mathsf{discard}\ \vec{y}\ \mathsf{in}\ N.$$

- (*Promotion*, *Contraction*)-cut.

$$\dfrac{\dfrac{!\Delta \vdash A}{!\Delta \vdash !A}\,Promotion \qquad \dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}}{\Gamma, !\Delta \vdash B}\,Cut$$

This reduces to

$$\dfrac{\dfrac{\dfrac{!\Delta \vdash A}{!\Delta \vdash !A}\,Prom. \qquad \dfrac{\dfrac{!\Delta \vdash A}{!\Delta \vdash !A}\,Prom. \qquad \Gamma, !A, !A \vdash B}{\Gamma, !\Delta, !A \vdash B}\,Cut}{\Gamma, !\Delta, !\Delta \vdash B}\,Cut}{\Gamma, !\Delta \vdash B}\,Contraction^{*}.$$

At the level of terms this gives the following cut-reduction

$$\begin{aligned} \mathsf{copy}\ (\mathsf{promote}\ \vec{y}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N)\ \mathsf{as}\ w, z\ \mathsf{in}\ P \quad \leadsto_{cut} \quad & \mathsf{copy}\ \vec{y}\ \mathsf{as}\ \vec{u}, \vec{v}\ \mathsf{in} \\ & P[\ \ w := \mathsf{promote}\ \vec{u}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N, \\ & \qquad z := \mathsf{promote}\ \vec{v}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N]. \end{aligned}$$

Thus the principal cuts correspond to the β-reduction rules from the natural deduction formulation. The only difference is where the left rule is either $\multimap$, & or *Dereliction*. In these cases we derive slightly more general reduction rules.

Secondary Cuts

There are 6 of these cuts and we shall consider them in turn.

- (*Promotion*, *Promotion*)-cut.

$$\dfrac{\dfrac{!\Delta \vdash A}{!\Delta \vdash !A}\,Promotion \qquad \dfrac{!\Gamma, !A \vdash B}{!\Gamma, !A \vdash !B}\,Promotion}{!\Gamma, !\Delta \vdash !B}\,Cut$$

This reduces to

$$\dfrac{\dfrac{!\Delta \vdash A}{!\Delta \vdash !A}\,\mathit{Promotion} \qquad !\Gamma, !A \vdash B}{\dfrac{!\Gamma, !\Delta \vdash B}{!\Gamma, !\Delta \vdash !B}\,\mathit{Promotion}}\,\mathit{Cut}$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{promote}\ \vec{w}, (\mathsf{promote}\ \vec{y}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{z}, u\ \mathsf{in}\ N \\ \leadsto_{cut}\ \mathsf{promote}\ \vec{w}, \vec{y}\ \mathsf{for}\ \vec{z}, \vec{t}\ \mathsf{in}\ N[u := \mathsf{promote}\ \vec{t}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ M]. \end{array}$$

- $(\otimes_{\mathcal{L}}, \mathit{Promotion})$-cut.

$$\dfrac{\dfrac{\Gamma, A, B \vdash !C}{\Gamma, A{\otimes}B \vdash !C}\,(\otimes_{\mathcal{L}}) \qquad \dfrac{!\Delta, !C \vdash D}{!\Delta, !C \vdash !D}\,\mathit{Promotion}}{\Gamma, !\Delta, A{\otimes}B \vdash !D}\,\mathit{Promotion.}$$

This reduces to

$$\dfrac{\dfrac{\Gamma, A, B \vdash !C \qquad \dfrac{!\Delta, !C \vdash D}{!\Delta, !C \vdash !D}\,\mathit{Promotion}}{\Gamma, !\Delta, A, B \vdash !D}\,\mathit{Cut}}{\Gamma, !\Delta, A{\otimes}B \vdash !D}\,(\otimes_{\mathcal{L}}).$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{promote}\ \vec{w}, (\mathsf{let}\ z\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N \\ \leadsto_{cut}\ \mathsf{let}\ z\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ (\mathsf{promote}\ \vec{w}, M\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N). \end{array}$$

- $(I_{\mathcal{L}}, \mathit{Promotion})$-cut.

$$\dfrac{\dfrac{\Gamma \vdash !A}{\Gamma, I \vdash !A}\,(I_{\mathcal{L}}) \qquad \dfrac{!\Delta, !A \vdash B}{!\Delta, !A \vdash !B}\,\mathit{Promotion}}{\Gamma, I, !\Delta \vdash !B}\,\mathit{Cut}$$

This reduces to

$$\dfrac{\dfrac{\Gamma \vdash !A \qquad \dfrac{!\Delta, !A \vdash B}{!\Delta, !A \vdash !B}\,\mathit{Promotion}}{\Gamma, !\Delta \vdash !B}\,\mathit{Cut}}{\Gamma, I, !\Delta \vdash !B}\,(I_{\mathcal{L}}).$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{promote}\ \vec{z}, (\mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N \\ \leadsto_{cut}\ \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{promote}\ \vec{z}, M\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N). \end{array}$$

- $(\oplus_{\mathcal{L}}, \mathit{Promotion})$-cut.

$$\dfrac{\dfrac{\Gamma, A \vdash\, !C \qquad \Gamma, B \vdash\, !C}{\Gamma, A \oplus B \vdash\, !C}\,(\oplus_{\mathcal{L}}) \qquad \dfrac{!\Delta, !C \vdash D}{!\Delta, !C \vdash\, !D}\,\mathit{Promotion}}{\Gamma, A \oplus B, !\Delta \vdash\, !D}\,\mathit{Cut}$$

This reduces to

$$\dfrac{\dfrac{\Gamma, A \vdash\, !C \qquad \dfrac{!\Delta, !C \vdash D}{!\Delta, !C \vdash\, !D}\,\mathit{Prom.}}{\Gamma, A, !\Delta \vdash\, !D}\,\mathit{Cut} \qquad \dfrac{\Gamma, B \vdash\, !C \qquad \dfrac{!\Delta, !C \vdash D}{!\Delta, !C \vdash\, !D}\,\mathit{Prom.}}{\Gamma, B, !\Delta \vdash\, !D}\,\mathit{Cut}}{\Gamma, A \oplus B, !\Delta \vdash\, !D}\,(\oplus_{\mathcal{L}}).$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{promote}\ \vec{w}, (\mathsf{case}\ z\ \mathsf{of}\ \mathsf{inl}(x) \ \to\ M \parallel \mathsf{inr}(y) \ \to\ N) \\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ P \\ \rightsquigarrow_{cut}\ \mathsf{case}\ z\ \mathsf{of} \\ \qquad\qquad \mathsf{inl}(x) \to \mathsf{promote}\ \vec{w}, M\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ P \\ \qquad\qquad \mathsf{inr}(y) \to \mathsf{promote}\ \vec{w}, N\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ P. \end{array}$$

- $(\mathit{Weakening}, \mathit{Promotion})$-cut.

$$\dfrac{\dfrac{\Gamma \vdash\, !B}{\Gamma, !A \vdash\, !B}\,\mathit{Weakening} \qquad \dfrac{!\Delta, !B \vdash C}{!\Delta, !B \vdash\, !C}\,\mathit{Promotion}}{\Gamma, !A, !\Delta \vdash\, !C}\,\mathit{Cut}$$

This reduces to

$$\dfrac{\dfrac{\Gamma \vdash\, !B \qquad \dfrac{!\Delta, !B \vdash C}{!\Delta, !B \vdash\, !C}\,\mathit{Promotion}}{\Gamma, !\Delta \vdash\, !C}\,\mathit{Cut}}{\Gamma, !A, !\Delta \vdash\, !C}\,\mathit{Weakening}.$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{promote}\ \vec{z}, (\mathsf{discard}\ x\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N \\ \rightsquigarrow_{cut}\ \mathsf{discard}\ x\ \mathsf{in}\ (\mathsf{promote}\ \vec{z}, M\ \mathsf{for}\ \vec{y}, u\ \mathsf{in}\ N). \end{array}$$

- $(\mathit{Contraction}, \mathit{Promotion})$-cut.

$$\dfrac{\dfrac{\Gamma, !A, !A \vdash\, !B}{\Gamma, !A \vdash\, !B}\,\mathit{Contraction} \qquad \dfrac{!\Delta, !B \vdash C}{!\Delta, !B \vdash\, !C}\,\mathit{Promotion}}{\Gamma, !A, !\Delta \vdash\, !C}\,\mathit{Cut}$$

This reduces to

$$\dfrac{\dfrac{\Gamma, !A, !A \vdash\, !B \qquad \dfrac{!\Delta, !B \vdash C}{!\Delta, !B \vdash\, !C}\,\mathit{Promotion}}{\Gamma, !A, !A, !\Delta \vdash\, !C}\,\mathit{Cut}}{\Gamma, !A, !\Delta \vdash C}\,\mathit{Contraction}.$$

At the level of terms this gives the cut-reduction

$$\begin{array}{l} \mathsf{promote}\ \vec{w}, (\mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ M)\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N \\ \rightsquigarrow_{cut}\ \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ (\mathsf{promote}\ \vec{w}, M\ \mathsf{for}\ \vec{v}, s\ \mathsf{in}\ N). \end{array}$$

These cuts we have seen before in the analysis of the subformula property of the natural deduction formulation. Seen from a natural deduction perspective they arise due to the fact that the *Promotion* rule introduces parasitic formulae and from a sequent calculus perspective that it not only introduces a connective on the right but imposes a strict condition on the left.

Commuting Cuts

There are 54 of these. It turns out that these are equivalent to the commuting conversions which arise from the natural deduction formulation (excluding those considered as secondary cuts and those classified as Unit Cuts). The term reductions are then as in Figures 3.7, 3.8 and 3.9 and we shall not repeat them here. As for the principal cut reductions, we find that some of the term reductions suggested by the cut-elimination process are actually more general than that considered as a commuting conversion. These more general rules are when the second left rule is either $\multimap_{\mathcal{L}}$, $\&_{\mathcal{L}-1}$, $\&_{\mathcal{L}-2}$ or *Dereliction.*

Unit Cuts

These are cuts of the form $(X, \mathbf{f}_{\mathcal{L}})$ and $(X, \mathbf{t}_{\mathcal{R}})$, for *all* rules, X. Thus there are 36 of these reduction rules. At the level of terms they can be expressed as two schemas, which are given in Figure 3.11. These match precisely the commuting conversions for the additive units given in Chapter 2.

$$\begin{array}{rcl} \mathsf{true}(M, \vec{x}) & \rightsquigarrow_{cut} & \mathsf{true}(\vec{y}, \vec{x}) \\ \mathsf{false}_A(M\ ;\ x) & \rightsquigarrow_{cut} & \mathsf{false}_A(\vec{y}\ ;\ x) \end{array}$$

where $FV(M) = \vec{y}$.

Figure 3.11: Unit Cut Reduction Rules

5 Properties of Reduction Rules

We shall consider two important properties for the reduction rules: *strong normalization* and *confluence* (Church-Rosser property). In particular, we shall only consider the β-reduction rules and omit the commuting conversions. This is not only because we would normally only consider using the β-reduction rules in any implementation, but also because of the explosion in complexity of considering the extra 102 commuting conversions! Given the uncertainty surrounding the additive units (Chapter 2, §1.2), they have not been included either. The technique we shall use is adapted from that used for the λ-calculus by Gallier [30], which is in turn adapted from proofs by Tait [74] and Girard [34]. The proof proceeds in two parts: Firstly, we give an alternative inductive definition of the linear terms (the so-called 'candidates') and then we show how the sets of strongly normalizing terms and confluent terms both satisfy this alternative inductive definition.

5.1 Candidates for the Linear Term Calculus

If a term is the result of an introduction rule, then if it is applied to another term then a redex could be formed. For the definition it is useful to divide terms up depending on whether they can introduce redexes or not.

Definition 17.

1. An *I-term* is a term of the form $M \otimes N$, $\lambda x\colon A.M$, $\langle M, N\rangle$, $\mathsf{inl}(M)$, $\mathsf{inr}(M)$ or $\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N$.
2. A *simple* term is one which is not an I-term.

3. A *stubborn* term, M, is a simple term which is either irreducible or if $M \rightsquigarrow_{\beta}^{+} N$ then N is a simple term.

Let $\mathcal{P} = \mathbf{P}_A$ be a family of nonempty sets of (typed) linear terms.

Definition 18. Properties (P1)–(P3) are defined as follows.

(P1) $x \in \mathbf{P}_A$ for every variable of type A.

(P2) If $M \in \mathbf{P}_A$ and $M \rightsquigarrow_{\beta} N$ then $N \in \mathbf{P}_A$.

(P3) If M is simple and

1. If $M \in \mathbf{P}_{A \multimap B}$, $N \in \mathbf{P}_A$ and $(\lambda x{:}A.M')N \in \mathbf{P}_B$ whenever $M \rightsquigarrow_{\beta}^{+} \lambda x{:}A.M'$ then $MN \in \mathbf{P}_B$.
2. If $M \in \mathbf{P}_{A\&B}$ and $\mathsf{fst}(\langle M', N'\rangle) \in \mathbf{P}_A$ and $\mathsf{snd}(\langle M', N'\rangle) \in \mathbf{P}_B$ whenever $M \rightsquigarrow_{\beta}^{+} \langle M', N'\rangle$ then $\mathsf{fst}(M) \in \mathbf{P}_A$ and $\mathsf{snd}(M) \in \mathbf{P}_B$.
3. If $M \in \mathbf{P}_{!A}$ and $\mathsf{derelict}(\mathsf{promote}\ \vec{P}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ Q) \in \mathbf{P}_A$ whenever $M \rightsquigarrow_{\beta}^{+} \mathsf{promote}\ \vec{P}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ Q$ then $\mathsf{derelict}(M) \in \mathbf{P}_A$.
4. If $M \in \mathbf{P}_{!A}$ and $Q \in \mathbf{P}_B$ and $\mathsf{discard}\ (\mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N)\ \mathsf{in}\ Q \in \mathbf{P}_B$ whenever $M \rightsquigarrow_{\beta}^{+} \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N$ then $\mathsf{discard}\ M\ \mathsf{in}\ Q \in \mathbf{P}_B$.
5. If $M \in \mathbf{P}_{!A}$ and $Q \in \mathbf{P}_B$ and $\mathsf{copy}\ (\mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N)\ \mathsf{as}\ y,z\ \mathsf{in}\ Q \in \mathbf{P}_B$ whenever $M \rightsquigarrow_{\beta}^{+} \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N$ then $\mathsf{copy}\ M\ \mathsf{as}\ y,z\ \mathsf{in}\ Q \in \mathbf{P}_B$.

Henceforth we only consider families $\mathcal{P}$ which satisfy conditions (P1)–(P3).

Definition 19. A non-empty set C of linear terms of type A is a $\mathcal{P}$*-candidate* iff it satisfies the following three conditions:

(R1) $C \subseteq \mathbf{P}_A$.

(R2) If $M \in C$ and $M \rightsquigarrow_{\beta} N$ then $N \in C$.

(R3) If M is simple, $M \in \mathbf{P}_A$ and $M' \in C$ whenever $M \rightsquigarrow_{\beta}^{+} M'$ and M' is an I-term, then $M \in C$.

Then given a family $\mathcal{P}$, for every type A, we define $[\![A]\!]$ as follows.

Definition 20.

$$[\![A]\!] \stackrel{\text{def}}{=} \mathbf{P}_A, \text{ where } A \text{ is atomic.}$$

$$[\![A \multimap B]\!] \stackrel{\text{def}}{=} \{M | M \in \mathbf{P}_{A \multimap B}, \text{ and for all } N, \text{ if } N \in [\![A]\!] \text{ then } MN \in [\![B]\!]\}$$

$$[\![A \otimes B]\!] \stackrel{\text{def}}{=} \left\{M | M \in \mathbf{P}_{A \otimes B}, M' \in [\![A]\!] \text{ and } N' \in [\![B]\!] \text{ whenever } M \rightsquigarrow_{\beta}^{*} M' \otimes N'\right\}$$

$$[\![I]\!] \stackrel{\text{def}}{=} \{M | M \in \mathbf{P}_I\}$$

$$[\![A \& B]\!] \stackrel{\text{def}}{=} \{M | M \in \mathbf{P}_{A\&B} \text{ and } \mathsf{fst}(M) \in [\![A]\!] \text{ and } \mathsf{snd}(M) \in [\![B]\!]\}$$

$$\begin{aligned}[\![A \oplus B]\!] \stackrel{\text{def}}{=}\ & \left\{M | M \in \mathbf{P}_{A \oplus B} \text{ and } M' \in [\![A]\!] \text{ whenever } M \rightsquigarrow_{\beta}^{*} \mathsf{inl}(M')\right\} \\ & \cap \left\{M | M \in \mathbf{P}_{A \oplus B} \text{ and } M'' \in [\![B]\!] \text{ whenever } M \rightsquigarrow_{\beta}^{*} \mathsf{inr}(M'')\right\}\end{aligned}$$

$$\begin{aligned}[\![!A]\!] \stackrel{\text{def}}{=}\ & \{M | M \in \mathbf{P}_{!A}, \mathsf{derelict}(M) \in [\![A]\!] \text{ and } \vec{M'} \in \vec{\mathbf{P}_{!B}} \text{ and } [\![N]\!] \in [\![A]\!] \\ & \quad \text{whenever } M \rightsquigarrow_{\beta}^{*} \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N\} \cap \\ & \{M | M \in \mathbf{P}_{!A}, \mathsf{discard}\ \vec{M'}\ \mathsf{in}\ Q \in \mathbf{P}_C \\ & \quad \text{whenever } M \rightsquigarrow_{\beta}^{*} \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N \text{ and } Q \in \mathbf{P}_C\ \} \cap \\ & \{M | M \in \mathbf{P}_{!A}, \\ & \ \mathsf{copy}\ \vec{M'}\ \mathsf{as}\ \vec{u}, \vec{v}\ \mathsf{in}\ R[y := \mathsf{promote}\ \vec{u}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N, z := \mathsf{promote}\ \vec{v}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N] \in \mathbf{P}_D \\ & \quad \text{whenever } M \rightsquigarrow_{\beta}^{*} \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N \text{ and } R \in \mathbf{P}_D\ \}\end{aligned}$$

Lemma 4. If $\mathcal{P}$ is a family satisfying (P1)–(P3), then each $[\![A]\!]$ is a $\mathcal{P}$-candidate which contains all the stubborn terms in $\mathbf{P}_A$.

Proof. We proceed by induction on the type A.

Case A atomic. (R1) holds trivially, (R2) holds from property (P2) and (R3) also holds trivially.

Case $A \multimap B$. Clearly (R1) holds by definition of $[\![A \multimap B]\!]$. To prove (R2) we assume that $M \in [\![A \multimap B]\!]$ and that $M \leadsto_\beta N$. From (R1) we have that $M \in \mathbf{P}_{A \multimap B}$. Thus by (P2) we have that $N \in \mathbf{P}_{A \multimap B}$. Let us take a $Q \in [\![A]\!]$. By the definition of $[\![A \multimap B]\!]$ we have that $MQ \in [\![B]\!]$. Since we have that $M \leadsto_\beta N$ then $MQ \leadsto_\beta NQ$ by the definition of reduction. By induction (at type B) we have that $NQ \in [\![B]\!]$. We can now conclude from the definition of $[\![A \multimap B]\!]$ that $N \in [\![A \multimap B]\!]$.

To prove (R3) we assume that M is simple, $M \in \mathbf{P}_{A \multimap B}$ and that $\lambda x{:}\, A.M' \in \mathbf{P}_{A \multimap B}$ whenever $M \leadsto_\beta^+ \lambda x{:}\, A.M'$. If M is stubborn then by induction at type B that $M \in [\![A \multimap B]\!]$ and we are done. If M is not stubborn then we know that $M \leadsto_\beta^+ \lambda x{:}\, A.M'$. We have by assumption that $\lambda x{:}\, A.M' \in [\![A \multimap B]\!]$. Thus taking an $N \in [\![A]\!]$, we have that $(\lambda x{:}\, A.M')N \in [\![B]\!]$. We also know that $N \in \mathbf{P}_{A \multimap B}$ and hence by (P3), $MN \in \mathbf{P}_B$. Now consider the reduction of MN. If MN is stubborn then we have by induction at type B that $MN \in [\![B]\!]$. If MN is not stubborn then its reduction path must be of the following form, where Q is an I-term and $N \leadsto_\beta^* N'$,

$$MN \leadsto_\beta^* (\lambda x{:}\, A.M')N' \leadsto_\beta M'[x := N'] \leadsto_\beta^* Q.$$

We know by (R2) that $N' \in [\![A]\!]$. Hence by definition we have that $(\lambda x{:}\, A.M')N' \in [\![B]\!]$. Again by (R2) we have that $Q \in [\![B]\!]$. Thus by induction at type B we have that $MN \in [\![B]\!]$ for any $N \in [\![B]\!]$. By definition we can conclude that $M \in [\![A \multimap B]\!]$ and we are done.

Case $A \otimes B$. Clearly (R1) holds by definition of $[\![A \otimes B]\!]$. To prove (R2) we shall assume that $M \in [\![A \otimes B]\!]$ and that $M \leadsto_\beta N$. We know from (R1) that $M \in \mathbf{P}_{A \otimes B}$ and, hence, by (P2), $N \in \mathbf{P}_{A \otimes B}$. By definition of $M \in [\![A \otimes B]\!]$, if $M \leadsto_\beta^* M' \otimes N'$ then $M' \in [\![A]\!]$ and that $N' \in [\![B]\!]$. It is clear that $M \leadsto_\beta^* M' \otimes N'$ whenever $N \leadsto_\beta^* M' \otimes N'$. Thus by definition $N \in [\![A \otimes B]\!]$.

To prove (R3) we assume that M is simple, $M \in \mathbf{P}_{A \otimes B}$ and that $M' \otimes N' \in \mathbf{P}_{A \otimes B}$ whenever $M \leadsto_\beta^* M' \otimes N'$. If M is stubborn then by the definition of $[\![A \otimes B]\!]$ we have that $M \in [\![A \otimes B]\!]$. If M is not stubborn then we have that $M \leadsto_\beta^+ M' \otimes N'$ and hence by (R2) we have that $M' \otimes N' \in [\![A \otimes B]\!]$. By the definition of $[\![A \otimes B]\!]$ we have that $M' \in [\![A]\!]$ and $N' \in [\![B]\!]$. Thus we can conclude that $M \in [\![A \otimes B]\!]$.

Case I. Clearly (R1) holds by definition of $[\![I]\!]$. To prove (R2) we assume that $M \in [\![I]\!]$ and that $M \leadsto_\beta N$. Thus we have that $M \in \mathbf{P}_I$ and then by (P2) that $N \in \mathbf{P}_I$. Clearly $M \leadsto_\beta^* *$ whenever $N \leadsto_\beta^* *$. We then have that $N \in [\![I]\!]$.

To prove (R3) we assume that M is simple and $M \in \mathbf{P}_I$. As M is always stubborn then trivially by definition we have that $M \in [\![I]\!]$

Case $A \& B$. Clearly (R1) holds by definition of $[\![A \& B]\!]$. To prove (R2) we assume that $N \in [\![A \& B]\!]$ and $M \leadsto_\beta N$. By (R1) we have that $M \in \mathbf{P}_{A \& B}$ and by (P2) we have that $N \in \mathbf{P}_{A \& B}$. By definition of $M \in [\![A \& B]\!]$ we have that $\mathsf{fst}(M) \in [\![A]\!]$ and $\mathsf{snd}(M) \in [\![B]\!]$. Since $\mathsf{fst}(M) \leadsto_\beta \mathsf{fst}(N)$ and $\mathsf{snd}(M) \leadsto_\beta \mathsf{snd}(N)$ we have by induction that $\mathsf{fst}(N) \in [\![A]\!]$ and $\mathsf{snd}(N) \in [\![B]\!]$. Hence we can conclude that $N \in [\![A \& B]\!]$.

To prove (R3) we assume that M is simple, $M \in \mathbf{P}_{A \& B}$ and that $\langle M', N' \rangle \in [\![A \& B]\!]$ whenever $M \leadsto_\beta^+ \langle M', N' \rangle$. If M is stubborn then clearly both $\mathsf{fst}(M)$ and $\mathsf{snd}(N)$ are stubborn. We know from (P3) that $\mathsf{fst}(M) \in \mathbf{P}_A$ and $\mathsf{snd}(M) \in \mathbf{P}_B$ and since by induction we have that both $[\![A]\!]$ and $[\![B]\!]$ contain all the stubborn terms in $\mathbf{P}_A$ and $\mathbf{P}_B$ respectively, then we have that $\mathsf{fst}(M) \in [\![A]\!]$ and $\mathsf{snd}(M) \in [\![B]\!]$. Thus we can conclude that $M \in [\![A \& B]\!]$. If M is not stubborn then we have that $M \rightsquigarrow_\beta^+ \langle M', N'\rangle$. By assumption we have that $\langle M', N'\rangle \in [\![A\&B]\!]$. Hence by definition we have that $\mathsf{fst}(\langle M', N'\rangle) \in [\![A]\!]$ and $\mathsf{snd}(\langle M', N'\rangle) \in [\![B]\!]$. By induction we also have that $\mathsf{fst}(\langle M', N'\rangle) \in \mathbf{P}_A$ and $\mathsf{snd}(\langle M', N'\rangle) \in \mathbf{P}_B$. Hence by (P3) we have that $\mathsf{fst}(M) \in \mathbf{P}_A$ and $\mathsf{snd}(M) \in \mathbf{P}_B$. Let us consider $\mathsf{fst}(M)$ (the case for $\mathsf{snd}(M)$ is symmetric). If $\mathsf{fst}(M)$ is stubborn then we have by induction that $\mathsf{fst}(M) \in [\![A]\!]$. If $\mathsf{fst}(M)$ is not stubborn then its reduction path must be of the following form, where Q is an I-term,

$$\mathsf{fst}(M) \rightsquigarrow_\beta^+ \mathsf{fst}(\langle M', N'\rangle) \rightsquigarrow_\beta M'' \rightsquigarrow_\beta^* Q.$$

Since $\mathsf{fst}(\langle M', N'\rangle) \in [\![A]\!]$ we have by induction that $Q \in [\![A]\!]$. Thus by induction we have that $\mathsf{fst}(M) \in [\![A]\!]$. By similar reasoning we deduce that $\mathsf{snd}(M) \in [\![B]\!]$ and hence by definition we can conclude $M \in [\![A\&B]\!]$.

Case $A \oplus B$. Clearly (R1) holds by definition of $[\![A \oplus B]\!]$. To prove (R2) we assume that $M \in [\![A \oplus B]\!]$ and that $M \rightsquigarrow_\beta N$. It is clear that $M \in \mathbf{P}_{A\oplus B}$ and from (P2) we have that $N \in \mathbf{P}_{A\oplus B}$. By definition of $M \in [\![A \oplus B]\!]$ we have that $M' \in [\![A]\!]$ whenever $M \rightsquigarrow_\beta^* \mathsf{inl}(M')$ and that $M'' \in [\![B]\!]$ whenever $M \rightsquigarrow_\beta^* \mathsf{inr}(M'')$. It is obvious that whenever $M \rightsquigarrow_\beta^* \mathsf{inl}(M')$ then $N \rightsquigarrow_\beta^* \mathsf{inl}(M')$ and that whenever $M \rightsquigarrow_\beta^* \mathsf{inr}(M'')$ then $N \rightsquigarrow_\beta^* \mathsf{inr}(M'')$. Hence we can conclude that $N \in [\![A \oplus B]\!]$ and we are done.

To prove (R3) we assume that M is simple, $M \in \mathbf{P}_{A\oplus B}$ and $M' \in [\![A\oplus B]\!]$ whenever $M \rightsquigarrow_\beta^+ M'$ and M' is an I-term. If M is a stubborn term then we have that it is never the case that $M \rightsquigarrow_\beta^+ \mathsf{inl}(M')$ nor $M \rightsquigarrow_\beta^+ \mathsf{inl}(M'')$, and so we conclude that $M \in [\![A \oplus B]\!]$. If M is not a stubborn term then we know that $M \rightsquigarrow_\beta^+ Q$ where Q is either $\mathsf{inl}(Q')$ or $\mathsf{inr}(Q'')$. Consider the first case. We have by assumption that $\mathsf{inl}(Q') \in [\![A \oplus B]\!]$. By definition we can conclude that $Q' \in [\![A]\!]$. Similarly we can conclude that $Q'' \in [\![B]\!]$. Thus we can conclude that $M \in [\![A \oplus B]\!]$ and we are done.

Case $!A$. Clearly (R1) holds by the definition of $[\![!A]\!]$. To prove (R2) we assume that $M \in [\![!A]\!]$ and that $M \rightsquigarrow_\beta Q$. We know that $M \in \mathbf{P}_{!A}$. Since $M \in [\![!A]\!]$ we know that a number of conditions hold when $M \rightsquigarrow_\beta^* \mathsf{promote}\, \vec{M'}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N$. But clearly whenever $M \rightsquigarrow_\beta^* \mathsf{promote}\, \vec{M'}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N$ it is the case that $Q \rightsquigarrow_\beta^* \mathsf{promote}\, \vec{M'}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N$. Thus we conclude that $Q \in [\![!A]\!]$.

To prove (R3) we assume that M is simple, $M \in \mathbf{P}_{!A}$ and that $\mathsf{promote}\, \vec{M}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N \in [\![!A]\!]$ whenever $M \rightsquigarrow_\beta^+ \mathsf{promote}\, \vec{M}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N$. If M is stubborn then the conditions of $[\![!A]\!]$ hold trivially and so we can conclude that $M \in [\![!A]\!]$. If M is not stubborn then we have that $M \rightsquigarrow_\beta^+ \mathsf{promote}\, \vec{M'}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N$ and by assumption we have that $\mathsf{promote}\, \vec{M'}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N \in [\![!A]\!]$. By definition we then have a number of conditions which hold when $\mathsf{promote}\, \vec{M'}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N \rightsquigarrow_\beta^* \mathsf{promote}\, \vec{M''}\, \mathsf{for}\, \vec{x''}\, \mathsf{in}\, N''$. In particular we can take the zero reduction case, and then we have the conditions sufficient to conclude that $M \in [\![!A]\!]$.

■

We can now extend the definition of $\mathcal{P}$.

Definition 21. Properties (P4) and (P5) are defined as follows.

(P4)
1. If $M \in \mathbf{P}_B$ and $x \in FV(M)$ then $\lambda x\colon A.M \in \mathbf{P}_{A\multimap B}$.
2. If $M \in \mathbf{P}_A$ and $N \in \mathbf{P}_B$ then $M \otimes N \in \mathbf{P}_{A\otimes B}$.
3. $* \in \mathbf{P}_I$.
4. If $M \in \mathbf{P}_A$ and $N \in \mathbf{P}_B$ then $\langle M, N\rangle \in \mathbf{P}_{A\&B}$.
5. If $M \in \mathbf{P}_A$ then $\mathsf{inl}(M) \in \mathbf{P}_{A\oplus B}$.
6. If $N \in \mathbf{P}_B$ then $\mathsf{inr}(N) \in \mathbf{P}_{A\oplus B}$.
7. If $N \in \mathbf{P}_A$ and $\vec{M} \in \vec{\mathbf{P}_{!B}}$ then $\mathsf{promote}\, \vec{M}\, \mathsf{for}\, \vec{x}\, \mathsf{in}\, N \in \mathbf{P}_{!A}$.

(P5)
1. If $N \in \mathbf{P}_A$ and $M[x := N] \in \mathbf{P}_B$ then $(\lambda x\colon A.M)N \in \mathbf{P}_B$.

2. If $M \in \mathbf{P}_I$, $N \in \mathbf{P}_A$ then $\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N \in \mathbf{P}_A$.
3. If $M \in \mathbf{P}_{A\otimes B}$ and $N[x := M', y := N'] \in \mathbf{P}_C$ whenever $M \leadsto_\beta^* M'\otimes N'$ then $\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ N \in \mathbf{P}_C$.
4. If $M \in \mathbf{P}_A$ and $N \in \mathbf{P}_B$ then $\mathsf{fst}(\langle M, N\rangle) \in \mathbf{P}_A$ and $\mathsf{snd}(\langle M, N\rangle) \in \mathbf{P}_B$.
5. If $Q \in \mathbf{P}_{A\oplus B}$, $M \in \mathbf{P}_C$, $N \in \mathbf{P}_C$ and $M[x := Q'] \in \mathbf{P}_C$ whenever $Q \leadsto_\beta^* \mathsf{inl}(Q')$ then $\mathsf{case}\ Q\ \mathsf{of}\ \mathsf{inl}(x) \to M \parallel \mathsf{inr}(y) \to N \in \mathbf{P}_C$.
6. If $Q \in \mathbf{P}_{A\oplus B}$, $M \in \mathbf{P}_C$, $N \in \mathbf{P}_C$ and $N[y := Q''] \in \mathbf{P}_C$ whenever $Q \leadsto_\beta^* \mathsf{inr}(Q'')$ then $\mathsf{case}\ Q\ \mathsf{of}\ \mathsf{inl}(x) \to M \parallel \mathsf{inr}(y) \to N \in \mathbf{P}_C$.
7. If $M \in \mathbf{P}_{!A}$ and $N[\vec{x} := \vec{M'}] \in \mathbf{P}_B$ whenever $M \leadsto_\beta^* \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N$ then $\mathsf{derelict}(M) \in \mathbf{P}_B$.
8. If $M \in \mathbf{P}_{!A}$, $Q \in \mathbf{P}_B$ and $\mathsf{discard}\ \vec{M'}\ \mathsf{in}\ Q \in \mathbf{P}_B$ whenever $M \leadsto_\beta^* \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N$ then $\mathsf{discard}\ M\ \mathsf{in}\ Q \in \mathbf{P}_B$.
9. If $M \in \mathbf{P}_{!A}$, $Q \in \mathbf{P}_B$ and

$$\mathsf{copy}\ \vec{M'}\ \mathsf{as}\ \vec{u}, \vec{v}\ \mathsf{in}\ Q[y := \mathsf{promote}\ \vec{u}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N, z := \mathsf{promote}\ \vec{v}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N] \in \mathbf{P}_B$$

   whenever $M \leadsto_\beta^* \mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N$ then $\mathsf{copy}\ M\ \mathsf{as}\ y, z\ \mathsf{in}\ Q \in \mathbf{P}_B$.

We can now prove some auxiliary facts.

Lemma 5. If $\mathcal{P}$ is a family satisfying conditions (P1)–(P5) then the following properties hold:

1. If for every N, ($N \in [\![A]\!]$ implies $M[x := N] \in [\![B]\!]$) then $\lambda x\colon A.M \in [\![A \multimap B]\!]$.
2. If $M \in [\![A\otimes B]\!]$ and for every N, Q, ($N \in [\![A]\!]$ and $Q \in [\![B]\!]$ implies $R[x := N, y := Q] \in [\![C]\!]$) then $\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ R \in [\![C]\!]$.
3. If $M \in [\![A \oplus B]\!]$ and for all $N \in [\![A]\!]$ such that $Q[x := N] \in [\![C]\!]$ and for all $N' \in [\![B]\!]$ such that $R[y := N'] \in [\![C]\!]$ then $\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to Q \parallel \mathsf{inr}(y) \to R \in [\![C]\!]$.

Proof. We tackle each case in turn. From Lemma 4 we have already that the sets of the form $[\![A]\!]$ have properties (R1)–(R3).

1. We know that for all variables $x \in [\![A]\!]$. By assumption we have that $M[x := x] \in [\![B]\!]$ and hence $M \in [\![B]\!]$. From lemma 4 we have that $M \in \mathbf{P}_A$. From (P4) we can conclude that $\lambda x\colon A.M \in \mathbf{P}_{A\multimap B}$. Let us take a $N \in [\![A]\!]$, thus $N \in \mathbf{P}_A$. By assumption we also have that $M[x := N] \in [\![B]\!]$ and hence $M[x := N] \in \mathbf{P}_B$. We have from (P5) that $(\lambda x\colon A.M)N \in \mathbf{P}_B$. If $(\lambda x\colon A.M)N$ is stubborn then we can conclude from (R3) that $(\lambda x\colon A.M)N \in [\![B]\!]$. If $(\lambda x\colon A.M)N$ is not stubborn then its reduction path must be of the following form, where Q is an I-term,

$$(\lambda x\colon A.M)N \leadsto_\beta^* (\lambda x\colon A.M')N' \leadsto_\beta M'[x := N'] \leadsto_\beta^* Q.$$

   We have that $M[x := N] \in [\![B]\!]$ and since $M[x := N] \leadsto_\beta^* M'[x := N'] \leadsto_\beta^* Q$ then by (R2) we have that $Q \in [\![B]\!]$. By (R3) we can conclude that $(\lambda x\colon A.M)N \in [\![B]\!]$. Thus we have have that for all $N \in [\![A]\!]$ that $(\lambda x\colon A.M)N \in [\![B]\!]$ and so we can conclude that $\lambda x\colon A.M \in [\![A \multimap B]\!]$.

2. By assumption we have that $M \in [\![A\otimes B]\!]$ and hence by lemma 4 $M \in \mathbf{P}_{A\otimes B}$. If we take two variables x and y, then we have that $x \in [\![A]\!]$ and $y \in [\![B]\!]$ and by assumption we have that $R[x := x, y := y] \in [\![C]\!]$; hence $R \in [\![C]\!]$. If $\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ R$ is stubborn then we have from (R3) that $\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ R \in [\![C]\!]$. If $\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ R$ is not stubborn then its reduction path must be of the following form, where T is an I-term,

$$\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ R \leadsto_\beta^* \mathsf{let}\ M'\otimes N'\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ R' \leadsto_\beta R'[x := M', y := N'] \leadsto_\beta^* T.$$

   Since we have that $M \in [\![A\otimes B]\!]$ then we know that $M' \in [\![A]\!]$ and $N' \in [\![B]\!]$. We know then by assumption that $R[x := M', y := N'] \in [\![C]\!]$ and hence by (R2) $R'[x := M', y := N'] \in [\![C]\!]$. We can then conclude that $\mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ N \in [\![C]\!]$.

3. We have by assumption that $M \in [\![A \oplus B]\!]$ and hence we have $M \in \mathbf{P}_{A\oplus B}$. Taking variables x and y we have that $Q[x := x] \in [\![C]\!]$ and $R[y := y] \in [\![C]\!]$. Hence we also have that $Q \in \mathbf{P}_C$ and $R \in \mathbf{P}_C$. If M is stubborn it is easy to see that $\mathsf{case}\, M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R$ is stubborn also. Also by (P4) we know that $\mathsf{case} M \mathsf{of} \mathsf{inl}(x) \to Q \| \mathsf{inr}(y) \to R \in \mathbf{P}_C$. Since we know that all the stubborn terms in $\mathbf{P}_C$ are in $[\![C]\!]$, we can see that $\mathsf{case} M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R \in [\![C]\!]$. If M is not stubborn then consider the case where $M \rightsquigarrow^+_\beta \mathsf{inl}(M')$. We have by (R2) that $\mathsf{inl}(M') \in [\![A \oplus B]\!]$. By defintion this gives us that $M' \in [\![A]\!]$ and hence $M' \in \mathbf{P}_A$. Thus from (P5) we can conclude that $\mathsf{case}\, M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R \in \mathbf{P}_C$.

   If $\mathsf{case}\, M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R$ is stubborn then from (R3) we can conclude that $\mathsf{case}\, M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R \in [\![C]\!]$. If $\mathsf{case}\, M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R$ is not stubborn then its reduction path must be of the following form, where T is an I-term,

$$\begin{array}{l}\mathsf{case}\, M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R \rightsquigarrow^*_\beta \\ \mathsf{case}\, \mathsf{inl}(M') \,\mathsf{of}\, \mathsf{inl}(x) \to Q' \,\|\, \mathsf{inr}(y) \to R' \rightsquigarrow_\beta Q'[x := M'] \rightsquigarrow^*_\beta T.\end{array}$$

   (and similarly if $M \rightsquigarrow^*_\beta \mathsf{inr}(M'')$.) Since $M \in [\![A \oplus B]\!]$ we have by definition that $M' \in [\![A]\!]$. We have by definition that $P[x := M'] \in [\![C]\!]$ and hence we have that $P'[x := M'] \in [\![C]\!]$. Performing the same reasoning for the case where $M \rightsquigarrow^*_\beta \mathsf{inr}(M'')$ enables us to conclude that $\mathsf{case}\, M \,\mathsf{of}\, \mathsf{inl}(x) \to Q \,\|\, \mathsf{inr}(y) \to R \in [\![C]\!]$

■

Lemma 6. If $\mathcal{P}$ is a family satisfying (P1)–(P5), then for every M of type A, for every substitution ϕ such that $\phi(y) \in [\![B]\!]$ for every $y{:}\,B \in FV(M)$, we have that $\phi(M) \in [\![A]\!]$.

Proof. By induction on the structure of M.

Case x. Trivial by the definition of ϕ.

Case MN. By induction we have that $\phi_1(M) \in [\![A \multimap B]\!]$ and $\phi_2(N) \in [\![A]\!]$. By the definition of $[\![A \multimap B]\!]$ we have that $\phi_1(M)\phi_2(N) \in [\![B]\!]$. Thus by definition of substitution $\phi(MN) \in [\![B]\!]$ where $\phi = \phi_1 \cup \phi_2$.

Case $\lambda x{:}\, A.M$. If we take a N such that $N \in [\![A]\!]$, then we can extend a substitution ϕ to $\phi\dagger[x \mapsto N]$. Using this extended substitution, by induction we have that $\phi'(M) \in [\![B]\!]$ which is equivalent to $\phi(M)[x := N] \in [\![B]\!]$. By Lemma 1 we have that $\lambda x{:}\, A.M \in [\![A \multimap B]\!]$.

Case $M \otimes N$. By induction we have that $\phi_1(M) \in [\![A]\!]$ and $\phi_2(N) \in [\![B]\!]$. Thus we have that $\phi_1(M) \in \mathbf{P}_A$ and $\phi_2(N) \in \mathbf{P}_B$ and hence by (P4) we have that $M \otimes N \in \mathbf{P}_{A\otimes B}$. If $M \otimes N$ is stubborn then we have by definition that $M \otimes N \in [\![A \otimes B]\!]$. If $M \otimes N$ is not stubborn then we have that $M \otimes N \rightsquigarrow^*_\beta M' \otimes N'$. We have from (R2) that $M' \in [\![A]\!]$ and $N' \in [\![B]\!]$. By definition we can conclude that $M \otimes N \in [\![A \otimes B]\!]$.

Case $\mathsf{let}\, M \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, N$. By induction we have that $\phi_1(M) \in [\![A \otimes B]\!]$. We also have by induction that $\phi_2(N) \in [\![C]\!]$. By considering $FV(N)$ it is clear that ϕ_2 is of the form $\phi_3 \dagger [x \mapsto P, y \mapsto Q]$ for $P \in [\![A]\!]$ and $Q \in [\![B]\!]$. Hence we have that $(\phi_3(N))[x := P, y := Q] \in [\![C]\!]$. By lemma 5 we have that $\mathsf{let}\, \phi_1(M) \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, \phi_3(N) \in [\![C]\!]$. By the definition of substitution we can conclude $\phi(\mathsf{let}\, M \,\mathsf{be}\, x \otimes y \,\mathsf{in}\, N) \in [\![C]\!]$ where $\phi = \phi_1 \cup \phi_3$.

Case $*$. From (P4) we have that $* \in \mathbf{P}_I$ and as $*$ is stubborn we have by (R3) that $* \in [\![I]\!]$ and we are done.

Case $\mathsf{let}\, M \,\mathsf{be}\, * \,\mathsf{in}\, N$. We have by induction that $\phi_1(M) \in [\![I]\!]$ and that $\phi_2(N) \in [\![A]\!]$. We have that $\phi_1(M) \in \mathbf{P}_I$ and $\phi_2(N) \in \mathbf{P}_A$ and then by (P5) we have that $\mathsf{let}\, \phi_1(M) \,\mathsf{be}\, * \,\mathsf{in}\, \phi_2(N) \in \mathbf{P}_A$. If $\mathsf{let}\, \phi_1(M) \,\mathsf{be}\, * \,\mathsf{in}\, \phi_2(N)$ is stubborn then we have trivially from (R3) that $\mathsf{let}\, \phi_1(M) \,\mathsf{be}\, * \,\mathsf{in}\, \phi_2(N) \in [\![A]\!]$. If it is not stubborn then its reduction path must be of the following form, where Q is an I-term,

$$\mathsf{let}\ \phi_1(M)\ \mathsf{be}\ *\ \mathsf{in}\ \phi_2(N) \leadsto_\beta^* \mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ N' \leadsto_\beta N' \leadsto_\beta^* Q.$$

Since $\phi_2(N) \in [\![A]\!]$ then by (R2) we have that $Q \in [\![A]\!]$ and hence by (R3) we can conclude that $\mathsf{let}\ \phi_1(M)\ \mathsf{be}\ *\ \mathsf{in}\ \phi_2(N) = \phi(\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N) \in [\![A]\!]$, where $\phi = \phi_1 \cup \phi_2$.

Case $\langle M, N\rangle$**.** We have by induction that $\phi(M) \in [\![A]\!]$ and $\phi N \in [\![B]\!]$ and by (R2) we have that $\phi(M) \in \mathbf{P}_A$ and $\phi(N) \in \mathbf{P}_B$. By (P4) we have that $\langle\phi(M), \phi(N)\rangle = \phi(\langle M, N\rangle) \in \mathbf{P}_{A\&B}$. We know from (P5) that $\mathsf{fst}(\phi(\langle M, N\rangle)) \in \mathbf{P}_A$ and that $\mathsf{snd}(\phi(\langle M, N\rangle)) \in \mathbf{P}_B$. Let us consider $\mathsf{fst}(\phi(\langle M, N\rangle))$ (the case for $\mathsf{snd}(\phi(\langle M, N\rangle))$ is similar). If $\mathsf{fst}(\phi(\langle M, N\rangle))$ is stubborn then we have trivially from (R3) that $\mathsf{fst}(\phi(\langle M, N\rangle)) \in [\![A]\!]$. If it is not stubborn then its reduction path must be of the following form, where Q is an I-term,

$$\mathsf{fst}(\phi(\langle M, N\rangle)) \leadsto_\beta^* \mathsf{fst}(\langle M', N'\rangle) \leadsto_\beta M' \leadsto_\beta^* Q.$$

Since we have that $\phi(M) \in [\![A]\!]$ we have by (R2) that $Q \in [\![A]\!]$. By (R3) we can conclude that $\mathsf{fst}(\phi(\langle M, N\rangle)) \in [\![A]\!]$. By similar reasoning we deduce that $\mathsf{snd}(\phi(\langle M, N\rangle)) \in [\![B]\!]$ and hence by definition we can conclude that $\phi(\langle M, N\rangle) \in [\![A\&B]\!]$.

Case $\mathsf{fst}(M)$**.** By induction we have that $\phi(M) \in [\![A\&B]\!]$. By definition we have that $\mathsf{fst}(\phi(M)) \in [\![A]\!]$ and hence $\phi(\mathsf{fst}(M)) \in [\![A]\!]$ and we are done.

Case $\mathsf{snd}(M)$**.** Similar to above case.

Case $\mathsf{inl}(M)$**.** We have by induction that $\phi(M) \in [\![A]\!]$. By (R1) we have that $\phi(M) \in \mathbf{P}_A$ and by (P4), $\mathsf{inl}(\phi(M)) = \phi(\mathsf{inl}(M)) \in \mathbf{P}_{A\oplus B}$. If $\phi(\mathsf{inl}(M))$ is stubborn then by (R3) trivially we have that $\phi(\mathsf{inl}(M)) \in [\![A \oplus B]\!]$. If it is not stubborn then its reduction path must be $\mathsf{inl}(\phi(M)) \leadsto_\beta^* \mathsf{inl}(M')$. By (R2) we have that $M' \in [\![A]\!]$ and hence we have by definition that $\mathsf{inl}(\phi(M)) \in [\![A \oplus B]\!]$.

Case $\mathsf{inr}(M)$**.** Similar to above case.

Case $\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to Q$**.** By induction we have that $\phi_1(M) \in [\![A \oplus B]\!]$ and that $\phi_2(N) \in [\![C]\!]$ and $\phi_3(Q) \in [\![C]\!]$ where $\phi_2 = \phi_4 \dagger [x \mapsto R]$ and $\phi_3 = \phi_4 \dagger [y \mapsto S]$. Thus we have that $(\phi_4(N))[x := R] \in [\![C]\!]$ and $(\phi_4(Q))[y := S] \in [\![C]\!]$. Then by lemma 5 we have that $\mathsf{case}\ \phi_1(M)\ \mathsf{of}\ \mathsf{inl}(x) \to \phi_4(N) \parallel \mathsf{inr}(y) \to \phi_4(Q) \in [\![C]\!]$ and thus we can conclude that $\phi(\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to Q) \in [\![C]\!]$ where $\phi = \phi_1 \cup \phi_4$.

Case $\mathsf{derelict}(M)$**.** We have by induction that $\phi(M) \in [\![!A]\!]$. By (R1) we have that $\phi(M) \in \mathbf{P}_{!A}$. If $\mathsf{derelict}(M)$ is stubborn then we have from (R3) trivially that $\mathsf{derelict}(M) \in [\![A]\!]$. If $\mathsf{derelict}(M)$ is not stubborn then its reduction path must be of the following form, where Q is an I-term,

$$\mathsf{derelict}(\phi(M)) \leadsto_\beta^* \mathsf{derelict}(\mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N) \leadsto_\beta N[\vec{x} := \vec{M'}] \leadsto_\beta^* Q.$$

By (R2) we have that $\mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N \in [\![!A]\!]$. Then by definition (and considering the zero reduction) we have that $\mathsf{derelict}(\mathsf{promote}\ \vec{M'}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N) \in [\![A]\!]$. Hence by (R2) we have that $Q \in [\![A]\!]$ and thus by (R3) we can conclude that $\mathsf{derelict}(\phi(M)) = \phi(\mathsf{derelict}(M)) \in [\![A]\!]$.

Case $\mathsf{discard}\ M\ \mathsf{in}\ N$**.** We have by induction that $\phi_1(M) \in [\![!A]\!]$ and $\phi_2(N) \in [\![B]\!]$. By (R1) we have that $\phi_1(M) \in \mathbf{P}_{!A}$ and $\phi_2(N) \in \mathbf{P}_B$. If $\phi_1(M)$ is stubborn then trivially from (P5) we have that $\mathsf{discard}\ \phi_1(M)\ \mathsf{in}\ \phi_2(N) \in \mathbf{P}_B$. If $\phi_1(M)$ is not stubborn then $\phi_1(M) \leadsto_\beta^* \mathsf{promote}\ \vec{P}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ R$. We have from (R2) that $\mathsf{promote}\ \vec{P}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ R \in [\![!A]\!]$. Hence by definition we have that $\mathsf{discard}\ \vec{P}\ \mathsf{in}\ \phi_2(N) \in \mathbf{P}_B$.

As $\mathsf{discard}\ \phi_1(M)\ \mathsf{in}\ \phi_2(N)$ is always stubborn, (R3) holds trivially and so we have that $\mathsf{discard}\ \phi_1(M)\ \mathsf{in}\ \phi_2(N) = \phi(\mathsf{discard}\ M\ \mathsf{in}\ N) \in [\![B]\!]$ where $\phi = \phi_1 \cup \phi_2$.

Case copy M as x, y in N. We have by induction that $\phi_1(M) \in [\![!A]\!]$ and $\phi_2(N) \in [\![B]\!]$. ϕ_2 is clearly of the form $\phi_3 \dagger [x \mapsto P, y \mapsto Q]$ where $P, Q \in [\![!A]\!]$. Thus we have that $(\phi_3(N))[x := P, y := Q] \in [\![B]\!]$. If $\phi_1(M)$ is stubborn then by (P3) we have that copy $\phi_1(M)$ as x, y in $\phi_3(N) \in \mathbf{P}_B$. If it is not stubborn then we have that $\phi_1(M) \leadsto_\beta^+$ promote $\vec{P}$ for $\vec{w}$ in R. By (R2) we have that promote $\vec{P}$ for $\vec{w}$ in $R \in [\![!A]\!]$. By definition we have that copy $\vec{P}$ as $\vec{u}, \vec{v}$ in $\phi_3(N)[x :=$ promote $\vec{u}$ for $\vec{v}$ in $R, y :=$ promote $\vec{u}$ for $\vec{v}$ in $R] \in \mathbf{P}_B$. By (P5) we can conclude that copy $\phi_1(M)$ as x, y in $\phi_2(N) \in \mathbf{P}_B$. As it is always stubborn then we have trivially from (R3) that copy $\phi_1(M)$ as x, y in $\phi_2(N) \in [\![B]\!]$. Thus we can conclude that $\phi($copy M as x, y in $N) \in [\![B]\!]$ where $\phi = \phi_1 \cup \phi_3$.

Case promote $\vec{M}$ for $\vec{x}$ in N. We have by induction that $\phi_1(\vec{M}) \in [\![!\vec{A}]\!]$ and that $\phi_2(N) \in [\![B]\!]$ We have by (R1) that $\phi_1(\vec{M}) \in \mathbf{P}_{!\vec{A}}$ and that $\phi_2(N) \in \mathbf{P}_B$. Thus from (P4) we can conclude that promote $\phi_1(\vec{M})$ for $\vec{x}$ in $N \in \mathbf{P}_{!B}$. If promote $\phi_1(\vec{M})$ for $\vec{x}$ in N is stubborn then we have from (R3) trivially that promote $\phi_1(\vec{M})$ for $\vec{x}$ in $N \in [\![!B]\!]$. If it is not stubborn then promote $\phi_1(\vec{M})$ for $\vec{x}$ in $N \leadsto_\beta^*$ promote $\vec{M'}$ for $\vec{x'}$ in N'. We have from (R2) that $N' \in [\![B]\!]$ and from (P2) that $\vec{M'} \in \mathbf{P}_{!\vec{A}}$. By taking the definition of $[\![!B]\!]$ we find that promote $\phi_1(\vec{M})$ for $\vec{x}$ in $N \in [\![!B]\!]$. Hence we can conclude that $\phi($promote $\vec{M}$ for $\vec{x}$ in $N) \in [\![!B]\!]$ where $\phi = \vec{\phi_1} \cup \phi_2$.

■

Theorem 17. If $\mathcal{P}$ is a family of linear terms satisfying conditions (P1)–(P5), then $\mathbf{P}_A = \Lambda_A$ for every type A.

Proof. Apply the lemma above to every term M of type A taking the identity substitution (which is valid since for all variables x of type B we have by (R3) that $x \in [\![B]\!]$). Thus $M \in [\![A]\!]$ for every term of type A, i.e. $\Lambda_A \subseteq \mathbf{P}_A$. Since it is obvious that $\mathbf{P}_A \subseteq \Lambda_A$, we can conclude that in fact $\mathbf{P}_A = \Lambda_A$. ■

5.2 Strong Normalization and Confluence

As we mentioned earlier, now we have an alternative definition of the linear terms we can use this to check the terms for the desired properties. Indeed the proofs turn out to be rather simple as all need to show is that the set of strongly normalizing and set of confluent terms both satisfy the definition of $\mathcal{P}$.

Theorem 18. The reduction relation, $\leadsto_\beta$, is strongly normalizing.

Proof. Let $\mathcal{P}$ be the family defined such that $\mathbf{P}_A = SN_A$ where SN_A is the set of strongly normalizing terms of type A. By Theorem 17 we simply have to show that this $\mathcal{P}$ satisfies the 23 conditions (P1)–(P5)! First we shall make the following observation which will facilitate the proof. Since there are only a finite number of redexes in a term, then its reduction tree must be finitely branching. If a term is strongly normalizing (SN), every path in its reduction tree is finite and by König's lemma we have that the tree is finite. For any SN term, M, we shall write $d(M)$ to denote the depth of its reduction tree. We can now check the conditions (P1)–(P5). As there are 23 conditions to check, we shall present only a few representative cases.

(P3)(1) We assume that M is simple, $M \in SN_{A \multimap B}$, $N \in SN_A$ and that $(\lambda x{:}A.M')N \in SN_B$ whenever $M \leadsto_\beta^+ \lambda x{:}A.M'$. Consider $MN \leadsto_\beta P$. It is either the case that $P = M_1 N$ where $M \leadsto_\beta M_1$ or $P = MN_1$ where $N \leadsto_\beta N_1$. Considering the former when M_1 is simple or the latter case, it is easy to see that by induction on $d(M) + d(N)$ we have that $P \in SN_B$ and hence that $MN \in SN_B$. If we take the former case where $M_1 = \lambda x{:}A.M'$ then we have by assumption that $P \in SN_B$ and hence we are done.

(P3)(3) We assume that M is simple, $M \in SN_{!A}$ and that derelict(promote $\vec{M'}$ for $\vec{x}$ in N) $\in SN_A$ whenever $M \leadsto_\beta^+$ promote $\vec{M'}$ for $\vec{x}$ in N. Let us consider derelict(M) $\leadsto_\beta P$. We have that either P = derelict(M_1) where M_1 is a simple term and that $M \leadsto_\beta M_1$ or that P = derelict(promote $\vec{M'}$ for $\vec{x}$ in N) where $M \leadsto_\beta$ promote $\vec{M'}$ for $\vec{x}$ in N. In the former case by induction on $d(M)$ we have that $P \in SN_A$. In the latter we have that $P \in SN_A$ by assumption and so we are done.

(P3)(4) We assume that M is simple, $M \in SN_{!A}$, $Q \in SN_B$ and that discard (promote $\vec{M'}$ for $\vec{x}$ in N) in $Q \in SN_B$ whenever $M \leadsto_\beta^+$ promote $\vec{M'}$ for $\vec{x}$ in N. Let us consider discard M in $Q \leadsto_\beta P$. We have three alternatives: Firstly, that P = discard M_1 in Q where $M \leadsto_\beta M_1$ and M_1 is simple. Secondly where P = discard M in Q_1 where $Q \leadsto_\beta Q_1$. Thirdly where P = discard (promote $\vec{M'}$ for $\vec{x}$ in N) in Q where $M \leadsto_\beta$ promote $\vec{M'}$ for $\vec{x}$ in N. By induction on $d(M) + d(Q)$ the first two cases give that $P \in SN_B$. Considering the latter, we have by assumption that $P \in SN_B$ and so we are done.

(P4)(1) We assume that $M \in SN_B$. Let us consider $\lambda A.M \leadsto_\beta P$. It is clear that $P = \lambda x{:}\,A.M_1$ where $M \leadsto_\beta M_1$. By a simple induction on $d(M)$ we have that $\lambda x{:}\,A.M \in SN_{A \multimap B}$.

All the other cases for (P4) hold in the same (trivial) way.

(P5)(1) We assume that $N \in SN_A$ and $M[x := N] \in SN_B$. Consider $(\lambda x{:}\,A.M)N \leadsto_\beta P$. We have three possibilities: firstly, where $P = (\lambda x{:}\,A.M_1)N$ where $M \leadsto_\beta M_1$; secondly where $P = (\lambda x{:}\,A.M)N_1$ where $N \leadsto_\beta N_1$ and thirdly where $P = M'[x := N]$ where $M = \lambda x{:}\,A.M'$. Considering the first two cases, by induction on $d(M) + d(N)$ we have that $P \in SN_B$. By assumption we have that for the third case $P \in SN_B$ and so we are done.

(P5)(7) We assume that $M \in SN_{!A}$ and that $N[\vec{x} := \vec{M'}] \in SN_B$ whenever $M \leadsto_\beta^*$ promote $\vec{M'}$ for $\vec{x}$ in N. Consider derelict(M) $\leadsto_\beta P$. Either P = derelict(M_1) where $M \leadsto_\beta M_1$; or $P = N[\vec{x} := \vec{M'}]$ where $M \leadsto_\beta$ promote $\vec{M'}$ for $\vec{x}$ in N. In the former case, by induction on $d(M)$ we have that $P \in SN_B$. In the latter case we have by induction that $P \in SN_B$ and we are done.

(P5)(8) We assume that $M \in SN_{!A}$, $Q \in SN_B$ and that discard $\vec{M'}$ in $Q \in SN_B$ whenever $M \leadsto_\beta^*$ promote $\vec{M'}$ for $\vec{x}$ in N. Let us consider discard M in $Q \leadsto_\beta P$. We have three possibilities: Firstly where P = discard M_1 in Q where $M \leadsto_\beta M_1$; secondly where P = discard M in Q_1 where $Q \leadsto_\beta Q_1$ and thirdly where P = discard $\vec{M'}$ in Q where M = promote $\vec{M'}$ for $\vec{x}$ in N. In the first two cases, by induction on $d(M) + d(Q)$ we have that $P \in SN_B$ and for the third case we have by assumption that $P \in SN_B$ and hence we are done.

■

Roorda [75, Chapter 21] has also proved strong normalization for (classical) linear logic using a technique proposed by Dragalin [26]. Benton [12] has proved strong normalization for the multiplicative, exponential fragment of the linear term calculus by devising an ingenious translation into terms of System F. This translation has the property that it preserves redexes and thus we can use the fact that System F is strongly normalizing to deduce strong normalization for the linear term calculus. We shall now consider the property of confluence.

Theorem 19. The reduction relation, $\leadsto_\beta$, is confluent.

Proof. Let $\mathcal{P}$ be the family defined such that $\mathbf{P}_A = CR_A$ where CR_A is the set of confluent terms of type A. Again all we have to show is that this $\mathcal{P}$ satisfies the 23 conditions (P1)–(P5). Again we shall content ourselves with only a few examples.

(P3)(1) We assume that M is simple, $M \in CR_{A \multimap B}$, $N \in CR_A$ and $(\lambda x{:}\,A.P)N \in CR_B$ whenever $M \leadsto_\beta^+ \lambda x{:}\,A.P$. Let us consider $MN \leadsto_\beta^* R$. It can be of two forms: Firstly $MN \leadsto_\beta^* M'N'$ where $M \leadsto_\beta^* M'$ and $N \leadsto_\beta^* N'$. We shall call this an 'independent' reduction. Secondly, it could be of the form $MN \leadsto_\beta^+ (\lambda x{:}\,A.P)N \leadsto_\beta^* (\lambda x{:}\,A.P')N' \leadsto_\beta P'[x := N'] \leadsto_\beta^* R$, which we shall call a 'top-level' reduction. We thus have 4 cases to consider.

1. Two independent reductions: $MN \leadsto_\beta^* M_1N_1$ and $MN \leadsto_\beta^* M_2N_2$. Since we have by assumption that both M and N are confluent, we then have a term M_3N_3 where $M_1 \leadsto_\beta^* M_3$, $M_2 \leadsto_\beta^* M_3$, $N_1 \leadsto_\beta^* N_3$ and $N_2 \leadsto_\beta^* N_3$ and then we are done.
2. Two top-level reductions:

$$MN \leadsto_\beta^+ (\lambda x\colon A.M_1)N \leadsto_\beta^* (\lambda x\colon A.M_1)N_1 \leadsto_\beta M_1[x := N_1] \leadsto_\beta^* R_1$$

and

$$MN \leadsto_\beta^+ (\lambda x\colon A.M_2)N \leadsto_\beta^* (\lambda x\colon A.M_2)N_2 \leadsto_\beta M_2[x := N_2] \leadsto_\beta^* R_2.$$

Since confluence holds for M, there is a M_3 such that $\lambda x\colon A.M_1 \leadsto_\beta^* M_3$ and $\lambda x\colon A.M_2 \leadsto_\beta^* M_3$. Hence we have that $(\lambda x\colon A.M_1)N \leadsto_\beta^* M_3N$ and $(\lambda x\colon A.M_1)N \leadsto_\beta^* R_1$. By assumption, then we have an R_3 such that $M_3N \leadsto_\beta^* R_3$ and $R_1 \leadsto_\beta^* R_3$. We also have that $(\lambda x\colon A.M_2)N \leadsto_\beta^* R_3$ and $(\lambda x\colon A.M_2)N \leadsto_\beta^* R_2$. Hence, also by assumption, there is a term R_4 such that $R_2 \leadsto_\beta^* R_4$ and $R_3 \leadsto_\beta^* R_4$. Thus we are done.
3. One independent and one to-level reduction:

$$MN \leadsto_\beta^+ (\lambda x\colon A.M_1)N \leadsto_\beta^* (\lambda x\colon A.M_1)N_1 \leadsto_\beta M_1[x := N_1] \leadsto_\beta^* R_1$$

and

$$MN \leadsto_\beta^* M_2N_2 = R_2.$$

As M and N are confluent by assumption, we have that there is a R_3 such that $(\lambda x\colon A.M_1)N_1 \leadsto_\beta^* R_3$ and $R_2 \leadsto_\beta^* R_3$. Again, by assumption, the term $(\lambda x\colon A.M_1)N_1$ is confluent, and thus there exists a term R_4 such that $R_1 \leadsto_\beta^* R_4$ and $R_3 \leadsto_\beta^* R_4$. Thus we are done.
4. The symmetric case to the above. (Omitted)

(P3)(3) We assume that M is simple, $M \in CR_{!A}$ and $\mathsf{derelict}(\mathsf{promote}\,\vec{P}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q) \in CR_{!A}$ whenever $M \leadsto_\beta^+ \mathsf{promote}\,\vec{P}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q$. Let us consider $\mathsf{derelict}(M) \leadsto_\beta^* R$; again, it can be an independent reduction: $\mathsf{derelict}(M) \leadsto_\beta^* \mathsf{derelict}(M')$ where $M \leadsto_\beta^* M'$, or it can be a top-level reduction where $\mathsf{derelict}(M) \leadsto_\beta^+ \mathsf{derelict}(\mathsf{promote}\,\vec{P}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q) \leadsto_\beta^* \mathsf{derelict}(\mathsf{promote}\,\vec{P'}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q') \leadsto_\beta Q'[\vec{x} := \vec{P'}] \leadsto_\beta^* R$. We have 4 cases to consider.

1. Two independent reductions: $\mathsf{derelict}(M) \leadsto_\beta^* \mathsf{derelict}(M_1)$ and $\mathsf{derelict}(M) \leadsto_\beta^* \mathsf{derelict}(M_2)$. As M is confluent by assumption, then there is a term M_3 such that $\mathsf{derelict}(M_1) \leadsto_\beta^* \mathsf{derelict}(M_3)$ and $\mathsf{derelict}(M_2) \leadsto_\beta^* \mathsf{derelict}(M_3)$ and we are done.
2. Two top-level reductions:

$$\begin{aligned}&\mathsf{derelict}(M) \leadsto_\beta^+ \mathsf{derelict}(\mathsf{promote}\,\vec{P_1}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_1)\\ &\leadsto_\beta^* \mathsf{derelict}(\mathsf{promote}\,\vec{P_1'}\,\mathsf{for}\,\vec{x_1}\,\mathsf{in}\,Q_1') \leadsto_\beta Q_1'[\vec{x} := \vec{P_1'}] \leadsto_\beta^* R_1\end{aligned}$$

and

$$\begin{aligned}&\mathsf{derelict}(M) \leadsto_\beta^+ \mathsf{derelict}(\mathsf{promote}\,\vec{P_2}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_2)\\ &\leadsto_\beta^* \mathsf{derelict}(\mathsf{promote}\,\vec{P_2'}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_2') \leadsto_\beta Q_2'[\vec{x} := \vec{P_2'}] \leadsto_\beta^* R_2.\end{aligned}$$

We have that M is confluent and so we have that there is a term R_3 such that $\mathsf{derelict}(\mathsf{promote}\,\vec{P_1}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_1) \leadsto_\beta^* \mathsf{derelict}(R_3)$ and $\mathsf{derelict}(\mathsf{promote}\,\vec{P_2}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_2) \leadsto_\beta^* \mathsf{derelict}(R_3)$. By assumption we than have that there exists a term R_4 such that $\mathsf{derelict}(R_3) \leadsto_\beta^* R_4$ and $R_1 \leadsto_\beta^* R_4$. Also given that $\mathsf{derelict}(\mathsf{promote}\,\vec{P_2}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_2)$ is also a confluent term by assumption, then there exists a term R_5 such that $R_4 \leadsto_\beta^* R_5$ and $R_2 \leadsto_\beta^* R_5$. Thus we are done.
3. One independent and one top-level reduction:

$$\begin{aligned}&\mathsf{derelict}(M) \leadsto_\beta^+ \mathsf{derelict}(\mathsf{promote}\,\vec{P_1}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_1)\\ &\leadsto_\beta^* \mathsf{derelict}(\mathsf{promote}\,\vec{P_1'}\,\mathsf{for}\,\vec{x}\,\mathsf{in}\,Q_1') \leadsto_\beta Q_1'[\vec{x} := \vec{P_1'}] \leadsto_\beta^* R_1\end{aligned}$$

and

$$\text{derelict}(M) \rightsquigarrow^*_\beta \text{derelict}(M_2) = R_2.$$

As M is confluent by assumption, we have that there is a term R_3 such that $R_2 \rightsquigarrow^*_\beta R_3$ and derelict(promote $\vec{P_1}$ for $\vec{x}$ in Q_1) $\rightsquigarrow^*_\beta R_3$. Again, by assumption, the term derelict(promote $\vec{P_1}$ for $\vec{x}$ in Q_1) is confluent, and thus there exists a term R_4 such that $R_1 \rightsquigarrow^*_\beta R_4$ and $R_3 \rightsquigarrow^*_\beta R_4$. Thus we are done.

4. The symmetric case to the above. (Omitted)

(P4)(1) We assume that $M \in CR_B$. If we have that $\lambda x{:}\, A.M \rightsquigarrow^*_\beta M_2$ and $\lambda x{:}\, A.M \rightsquigarrow^*_\beta M_3$, then it must be the case that $M_2 = \lambda x{:}\, A.M_2'$ and $M_3 = \lambda x{:}\, A.M_3'$ where $M \rightsquigarrow^*_\beta M_2'$ and $M \rightsquigarrow^*_\beta M_3'$. By assumption there exists a term M_4 such that $M_2 \rightsquigarrow^*_\beta \lambda x{:}\, A.M_4$ and $M_3 \rightsquigarrow^*_\beta \lambda x{:}\, A.M_4$ and hence we are done. All other cases for (P4) are similar.

(P5)(8) We assume that $M \in CR_{!A}$, $Q \in CR_B$ and that discard $\vec{P}$ in $Q \in CR_B$ whenever $M \rightsquigarrow^*_\beta$ promote $\vec{P}$ for $\vec{x}$ in Q. Again we have two sorts of reductions and hence 4 cases to consider.

1. Two independent reductions: discard M in $Q \rightsquigarrow^*_\beta$ discard M_1 in Q_1 and discard M_2 in Q_2. Since we have assumption that both M and Q are confluent, then there are terms M_3 and Q_3 such that discard M_1 in Q_1 $\rightsquigarrow^*_\beta$ discard M_3 in Q_3 and discard M_2 in Q_2 $\rightsquigarrow^*_\beta$ discard M_3 in Q_3 and so we are done.
2. Two top-level reductions:

$$\text{discard } M \text{ in } Q \rightsquigarrow^*_\beta \text{discard (promote } \vec{P_1} \text{ for } \vec{x} \text{ in } N_1) \text{ in } Q_1 \rightsquigarrow_\beta \text{discard } \vec{P_1} \text{ in } Q_1 \rightsquigarrow^*_\beta R_1$$

and

$$\text{discard } M \text{ in } Q \rightsquigarrow^*_\beta \text{discard (promote } \vec{P_2} \text{ for } \vec{x} \text{ in } N_2) \text{ in } Q_2 \rightsquigarrow_\beta \text{discard } \vec{P_2} \text{ in } Q_2 \rightsquigarrow^*_\beta R_2.$$

As M and Q are both confluent by assumption, we have that there exists a term, discard M_2 in Q_3, such that discard (promote $\vec{P_1}$ for $\vec{x}$ in N_1) in $Q_1 \rightsquigarrow^*_\beta$ discard M_2 in Q_3 and discard (promote $\vec{P_2}$ for $\vec{x}$ in N_2) in Q_2 $\rightsquigarrow^*_\beta$ discard M_2 in Q_3. In fact, M_2 must be of the form promote $\vec{P_3}$ for $\vec{x}$ in N_3. Thus we have that discard (promote $\vec{P_3}$ for $\vec{x}$ in N_3) in $Q_3 \rightsquigarrow_\beta$ discard $\vec{P_3}$ in Q_3. We have by assumption that discard $\vec{P_1}$ in Q_1 is confluent, and so there exists a term R_3 such that $R_1 \rightsquigarrow^*_\beta R_3$ and discard $\vec{P_3}$ in Q_3. We also have by assumption that discard $\vec{P_2}$ in Q_2 is confluent and so there exists a term R_4 such that $R_3 \rightsquigarrow^*_\beta R_4$ and $R_2 \rightsquigarrow^*_\beta R_4$. Thus we are done.
3. One independent and one top-level reduction:

$$\text{discard } M \text{ in } Q \rightsquigarrow^*_\beta \text{discard (promote } \vec{P_1} \text{ for } \vec{x} \text{ in } N_1) \text{ in } Q_1 \rightsquigarrow_\beta \text{discard } \vec{P_1} \text{ in } Q_1 \rightsquigarrow^*_\beta R_1$$

and

$$\text{discard } M \text{ in } Q \rightsquigarrow^*_\beta \text{discard } M_2 \text{ in } Q_2 = R_2.$$

As M and Q are both confluent by assumption, we have that there exists a term, discard M_3 in Q_3, such that discard (promote $\vec{P_1}$ for $\vec{x}$ in N_1) in Q_1 $\rightsquigarrow^*_\beta$ discard M_3 in Q_3 and discard M_2 in Q_2 $\rightsquigarrow^*_\beta$ discard M_3 in Q_3. In fact M_3 must be of the form promote $\vec{P_2}$ for $\vec{x}$ in N_2. Thus we have that discard (promote $\vec{P_2}$ for $\vec{x}$ in N_2) in Q_3 $\rightsquigarrow_\beta$ discard $\vec{P_2}$ in Q_3. We have by assumption that discard $\vec{P_1}$ in Q_1 is confluent and so there exists a term R_3 such that $R_1 \rightsquigarrow^*_\beta R_3$ and discard $\vec{P_3}$ in $Q_3 \rightsquigarrow^*_\beta R_3$. Also by assumption, discard $\vec{P_2}$ in Q_2 is confluent and so there exists a term R_4 such that $R_2 \rightsquigarrow^*_\beta R_4$ and $R_3 \rightsquigarrow^*_\beta R_4$. Thus we are done.
4. The symmetric case to the above. (Omitted)

■

6 Compilation into Linear Combinators

In §4.3 of Chapter 2 we gave a procedure for translating deductions from the natural deduction to the axiomatic formulation. By using the Curry-Howard correspondence, we can express this as a translation from terms of the linear term calculus to linear combinators. (As we mentioned earlier, this can be thought of as an implementation technique in a similar way to that proposed by Turner for the λ-calculus.) We give the resulting translation procedure, $[\![-]\!]_C$, in Figure 3.12.

$$
\begin{array}{rcl}
[\![x]\!]_C & \stackrel{\text{def}}{=} & x \\
[\![\lambda x{:}\,A.M]\!]_C & \stackrel{\text{def}}{=} & [x]\,[\![M]\!]_C \\
[\![MN]\!]_C & \stackrel{\text{def}}{=} & [\![M]\!]_C\,[\![N]\!]_C \\
[\![*]\!]_C & \stackrel{\text{def}}{=} & \texttt{unit} \\
[\![\mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N]\!]_C & \stackrel{\text{def}}{=} & (\texttt{let}\,[\![N]\!]_C)\,[\![M]\!]_C \\
[\![M{\otimes}N]\!]_C & \stackrel{\text{def}}{=} & (\texttt{tensor}\,[\![M]\!]_C)\,[\![N]\!]_C \\
[\![\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N]\!]_C & \stackrel{\text{def}}{=} & (\texttt{split}\,[\![M]\!]_C)\,([x]([y]\,[\![N]\!]_C)) \\
[\![\langle M,N\rangle]\!]_C & \stackrel{\text{def}}{=} & \texttt{with}([\![M]\!]_C,[\![N]\!]_C) \\
[\![\mathsf{fst}(M)]\!]_C & \stackrel{\text{def}}{=} & \texttt{fst}\,[\![M]\!]_C \\
[\![\mathsf{snd}(M)]\!]_C & \stackrel{\text{def}}{=} & \texttt{snd}\,[\![M]\!]_C \\
[\![\mathsf{inl}(M)]\!]_C & \stackrel{\text{def}}{=} & \texttt{inl}\,[\![M]\!]_C \\
[\![\mathsf{inr}(M)]\!]_C & \stackrel{\text{def}}{=} & \texttt{inr}\,[\![M]\!]_C \\
[\![\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P]\!]_C & \stackrel{\text{def}}{=} & \texttt{sdist}\,(\texttt{with}([x][\![N]\!]_C,[y][\![P]\!]_C))\,[\![M]\!]_C \\
[\![\mathsf{true}(\vec{M})]\!]_C & \stackrel{\text{def}}{=} & \texttt{term}\,(\texttt{tensor}\ldots \\
 & & \qquad \texttt{tensor}\,[\![M_1]\!]_C\,[\![M_2]\!]_C\ldots[\![M_n]\!]_C) \\
[\![\mathsf{false}_A(\vec{M}\,;\,N)]\!]_C & \stackrel{\text{def}}{=} & \texttt{init}\,[\![N]\!]_C\,[\![M_1]\!]_C\ldots[\![M_n]\!]_C \\
[\![\mathsf{discard}\ M\ \mathsf{in}\ N]\!]_C & \stackrel{\text{def}}{=} & (\texttt{disc}\,[\![N]\!]_C)\,[\![M]\!]_C \\
[\![\mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N]\!]_C & \stackrel{\text{def}}{=} & (\texttt{dupl}\,([x]([y]\,[\![N]\!]_C)))\,[\![M]\!]_C \\
[\![\mathsf{derelict}(M)]\!]_C & \stackrel{\text{def}}{=} & \texttt{eps}\,[\![M]\!]_C \\
[\![\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N]\!]_C & \stackrel{\text{def}}{=} & \mathrm{acc}(\texttt{promote}([\vec{x}][\![N]\!]_C),[\![\vec{M}]\!]_C)
\end{array}
$$

where
$$
\begin{array}{rcl}
\mathrm{acc}(e,[f_1,f_2,\ldots]) & \stackrel{\text{def}}{=} & \mathrm{acc}((\texttt{B}\,(\texttt{edist}\,e)\,\texttt{delta}\,f_1),[f_2,\ldots]) \\
\mathrm{acc}(e,[\,]) & \stackrel{\text{def}}{=} & e
\end{array}
$$

Figure 3.12: Compiling Linear Terms into Linear Combinators

We can also give a translation function, $[\![-]\!]_\lambda$, in the other direction, namely converting combinatory terms into linear terms. This is given in Figure 3.13.

As we might have expected, these translation functions have the desired property in that they map valid linear terms to valid combinator terms and vice versa.

Lemma 7.

1. If $\vec{x}{:}\,\Gamma \triangleright M{:}\,A$ then $\vec{x}{:}\,\Gamma \Rightarrow [\![M]\!]_C{:}\,A$.
2. If $\vec{x}{:}\,\Gamma \Rightarrow e{:}\,A$ then $\vec{x}{:}\,\Gamma \triangleright [\![e]\!]_\lambda{:}\,A$.

Proof. By structural induction. ∎

We did not mention reduction for the axiomatic formulation in Chapter 2, not least because it seems hard to motivate unless considered within the context of an equivalent term calculus. Indeed, to the best of my knowledge, there does not seem to be any natural method for deducing the reduction

$$
\begin{array}{rcl}
[\![\mathtt{I}_A]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : A.x \\
[\![\mathtt{B}_{A,B,C}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda g : (B{-\!\circ}C).\lambda f : (A{-\!\circ}B).\lambda x : A.fgx \\
[\![\mathtt{C}_{A,B,C}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda f : (A{-\!\circ}(B{-\!\circ}C)).\lambda x : B.\lambda y : A.fyx \\
\\
[\![\mathtt{tensor}_{A,B}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : A.\lambda y : B.x{\otimes}y \\
[\![\mathtt{split}_{A,B,C}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : A{\otimes}B.\lambda f : (A{-\!\circ}(B{-\!\circ}C)).\mathsf{let}\ x\ \mathsf{be}\ y{\otimes}z\ \mathsf{in}\ fyz \\
\\
[\![\mathtt{unit}]\!]_\lambda & \stackrel{\text{def}}{=} & * \\
[\![\mathtt{let}_A]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : A.\lambda y : I.\mathsf{let}\ y\ \mathsf{be}\ *\ \mathsf{in}\ x \\
\\
[\![\mathtt{fst}_{A,B}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : A\&B.\mathsf{fst}(x) \\
[\![\mathtt{snd}_{A,B}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : A\&B.\mathsf{snd}(x) \\
[\![\mathtt{wdist}_{A,B,C}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda f : ((A{-\!\circ}B)\&(A{-\!\circ}C)).\lambda y : A.\langle(\mathsf{fst}(f)y),(\mathsf{snd}(f)y)\rangle \\
\\
[\![\mathtt{inl}_{A,B}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : A.\mathsf{inl}(x) \\
[\![\mathtt{inr}_{A,B}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda y : B.\mathsf{inr}(y) \\
[\![\mathtt{sdist}_{A,B,C}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda f : ((A{-\!\circ}C)\&(B{-\!\circ}C)).\lambda x{:}\, A \oplus B. \\
 & & \quad \mathsf{case}\ x\ \mathsf{of} \\
 & & \quad \mathsf{inl}(x) \to (\mathsf{fst}(f))x \parallel \mathsf{inr}(y) \to (\mathsf{snd}(f))y \\
\\
[\![\mathtt{init}_A]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x{:}\, \mathbf{f}.\mathsf{false}_A(\_;\, x) \\
[\![\mathtt{term}_A]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x{:}\, A.\mathsf{true}(x) \\
\\
[\![\mathtt{dupl}_A]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda f : (!A{-\!\circ}(!A{-\!\circ}B)).\lambda x :!A.\mathsf{copy}\ x\ \mathsf{as}\ y,z\ \mathsf{in}\ fyz \\
[\![\mathtt{disc}_{A,B}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x : B.\lambda y :!A.\mathsf{discard}\ y\ \mathsf{in}\ x \\
[\![\mathtt{eps}_A]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x :!A.\mathsf{derelict}(x) \\
[\![\mathtt{delta}_A]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda x :!A.\mathsf{promote}\ x\ \mathsf{for}\ y\ \mathsf{in}\ y \\
[\![\mathtt{edist}_{A,B}]\!]_\lambda & \stackrel{\text{def}}{=} & \lambda f :!(A{-\!\circ}B).\lambda x :!A.\mathsf{promote}\ f,x\ \mathsf{for}\ g,y\ \mathsf{in}\ (\mathsf{derelict}(g)\mathsf{derelict}(y)) \\
\\
[\![x]\!]_\lambda & \stackrel{\text{def}}{=} & x \\
[\![ef]\!]_\lambda & \stackrel{\text{def}}{=} & [\![e]\!]_\lambda\, [\![f]\!]_\lambda \\
[\![\mathtt{with}(e,f)]\!]_\lambda & \stackrel{\text{def}}{=} & \langle([\![e]\!]_\lambda),([\![f]\!]_\lambda)\rangle \\
[\![\mathtt{promote}(e)]\!]_\lambda & \stackrel{\text{def}}{=} & \mathsf{promote}\ \_\ \mathsf{for}\ \_\ \mathsf{in}\ [\![e]\!]_\lambda
\end{array}
$$

Figure 3.13: Compiling Linear Combinators into Linear Terms

rules. As we used the natural deduction formulation to deduce the set of combinators initially, it seems appropriate to use the equivalence between linear terms and linear combinators along with the β-reduction relation for linear terms to determine the reduction rules for the linear combinatory terms. The resulting reduction relation, $\leadsto_w$, (where the w refers to the 'weak' sense of reduction) is defined in Figure 3.14.

$$
\begin{array}{rcl}
\mathtt{I}\, e & \leadsto_w & e \\
\mathtt{B}\, e\, f\, g & \leadsto_w & e\,(f\, g) \\
\mathtt{C}\, e\, f\, g & \leadsto_w & (e\, g)\, f \\
\\
\mathtt{split}\,(\mathtt{tensor}\, e\, f)\, g & \leadsto_w & g\, e\, f \\
\mathtt{let}\, e\, \mathtt{unit} & \leadsto_w & e \\
\\
\mathtt{fst}\,(\mathtt{with}(e,f)) & \leadsto_w & e \\
\mathtt{snd}\,(\mathtt{with}(e,f)) & \leadsto_w & f \\
\mathtt{wdist}\,(\mathtt{with}(e,f))\, g & \leadsto_w & \mathtt{with}(eg,fg) \\
\\
\mathtt{sdist}\,(\mathtt{with}(e,f))\,(\mathtt{inl}\, g) & \leadsto_w & eg \\
\mathtt{sdist}\,(\mathtt{with}(e,f))\,(\mathtt{inr}\, g) & \leadsto_w & fg \\
\\
\mathtt{eps}\,(\mathtt{promote}(e)) & \leadsto_w & e \\
\mathtt{delta}\,(\mathtt{promote}(e)) & \leadsto_w & \mathtt{promote}(\mathtt{promote}(e)) \\
\mathtt{edist}\,(\mathtt{promote}(e))\,(\mathtt{promote}(f)) & \leadsto_w & \mathtt{promote}(ef) \\
\mathtt{disc}\, e\,(\mathtt{promote}(f)) & \leadsto_w & e \\
\mathtt{dupl}\, e\,(\mathtt{promote}(f)) & \leadsto_w & e\,(\mathtt{promote}(f))\,(\mathtt{promote}(f))
\end{array}
$$

Figure 3.14: Weak Reduction for Combinatory Terms

We have that this reduction relation reduces valid (combinatory) terms to valid terms. This property is related to subject reduction for the linear term calculus.

Lemma 8. If $\vec{x}{:}\Gamma \Rightarrow e{:}A$ and $e \leadsto_w f$ then $\vec{x}{:}\Gamma \Rightarrow f{:}A$.

Proof. By induction on the structure of e. ■

In addition we have some further properties of linear combinators which we shall find useful.[5]

Lemma 9.

1. $FV([x]e) = FV(e) - \{x\}$
2. $([x]e)x \leadsto_w^* e$
3. $([x]e)f \leadsto_w^* e[x := f]$

Proof. All by induction on the linear combinatory term. ■

We can now consider the similarity between reduction for the linear combinatory terms and for the linear terms. As is the case for the λ-calculus [39, Chapter 9], one direction is easy.

Lemma 10. If $e \leadsto_w f$ then $[\![e]\!]_\lambda \leadsto_\beta [\![f]\!]_\lambda$.

[5]We defined the set of free variables of a linear combinatory term (denoted by $FV(e)$) in Figure 3.5.

Proof. By induction on the structure of e. ■

The other direction, that β-reduction is mirrored by w-reduction, fails for the λ-calculus (and is why reduction for combinatory logic is referred to as 'weak'). Our refined use of **ILL** gives a glimmer of hope for this problem. For a fragment of the term calculus (the non-exponential fragment) we find that the required lemma *does* hold.

Lemma 11. Let M be a *non-exponential* term. If $M \rightsquigarrow_\beta N$ then $[\![M]\!]_C \rightsquigarrow_w^* [\![N]\!]_C$.

Proof. By induction on the structure of the linear term, M. ■

However, as for the λ-calculus [39], we still do not have an immediate relationship between the respective notions of normal form. This is a serious problem and one that has, by and large, been ignored by the functional programming community (for one), who use combinatory reduction to implement β-reduction in compilers [71, 78]. Despite our use of the refined setting of **ILL**, so far we gain no new insight into this problem; this remains an important open question.

Proposition 5. If e is in w-normal form, then it is *not* necessarily the case that $[\![e]\!]_\lambda$ is in β-normal form.

Proof. (Counterexample) Consider the combinatory term `let` e. It is in w-normal form, but its translation $(\lambda x.\lambda y.\mathsf{let}\ y\ \mathsf{be}\ *\ \mathsf{in}\ x)[\![e]\!]_\lambda$ is clearly not in β-normal form. ■

7 Translations

In §5 of Chapter 2 we considered the logical power of **ILL**. In particular, we saw how the exponential regained the power of **IL** and to that purpose we detailed a translation procedure, originally due to Girard [31], from a natural deduction formulation of **IL** into the natural deduction formulation of **ILL**. We can apply the Curry-Howard correspondence to this translation procedure to derive a translation from the extended[6] λ-calculus to the linear term calculus. First of all let us recall the natural deduction formulation of the extended λ-calculus in Figure 3.15.

We shall denote the term translation function as $|-|^\circ$. However, the translation is really on derivations rather than terms. There is a slight problem for the extended lambda calculus in that there is no syntax for the structural rules, in particular the rule of *Weakening.* Thus there is no difference between the term associated with the proof of $\Gamma \vdash A$ and one of $\Gamma, \Delta \vdash A$ which differs only by its instances of the *Id* rule (where the *Weakening* rule is embedded). This is a fault of the extended λ-calculus: the terms do *not* encode precisely the proof trees. Thus we shall include the context with the translation function, thus $|-|^\circ_\Gamma$. We give the translation procedure in Figure 3.16.

It should be noted that in some places α-conversions have been included. This is to ensure that the free variables of the original and the translated term are the same. As we might expect the translation maps valid λ-terms to valid linear terms.

Theorem 20. If $\vdash_{IL} \vec{x}{:}\Gamma \rhd M{:}A$ then $\vdash_{ILL} \vec{x}{:}!(\Gamma^\circ) \rhd |M|^\circ_\Gamma{:}A^\circ$.

Proof. By structural induction and use of Figure 3.16. ■

Let us return to the question raised in Chapter 2 concerning the preservation of normal forms by the Girard translation. Take the λ-term $\lambda f{:}A \supset B.\lambda x{:}A.fx$. Clearly this term is in (β, c)-normal form. However, consider its translation into a linear term:

$$\begin{array}{l}\lambda f{:}!(!A \multimap B).\lambda x{:}!A.\mathsf{copy}\ f, x\ \mathsf{as}\ f', f'', x', x''\\ \mathsf{in}\\ \underline{(\mathsf{discard}\ x'\ \mathsf{in}\ \mathsf{derelict}(f'))(\mathsf{promote}\ f'', x''\ \mathsf{for}\ f, x\ \mathsf{in}\ (\mathsf{discard}\ f\ \mathsf{in}\ \mathsf{derelict}(x)))}\end{array}$$

[6]The phrase 'λ-calculus' normally applies just to the implication fragment. We shall use the prefix 'extended' to refer to the term calculus corresponding to the whole of **IL**.

$$\frac{}{\Gamma, x{:}\,A \rhd x{:}\,A}\;Id$$

$$\frac{\Gamma \rhd M{:}\,\bot}{\Gamma \rhd \nabla_A(M){:}\,A}\;(\bot_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M{:}\,A \qquad \Gamma \rhd N{:}\,B}{\Gamma \rhd \langle M, N\rangle{:}\,A \wedge B}\;(\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M{:}\,A \wedge B}{\Gamma \rhd \mathsf{fst}(M){:}\,A}\;(\wedge_{\mathcal{E}-1}) \qquad \frac{\Gamma \rhd M{:}\,A \wedge B}{\Gamma \rhd \mathsf{snd}(M){:}\,B}\;(\wedge_{\mathcal{E}-2})$$

$$\frac{\Gamma, x{:}\,A \rhd M{:}\,B}{\Gamma \rhd \lambda x{:}\,A.M{:}\,A \supset B}\;(\supset_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M{:}\,A \supset B \qquad \Gamma \rhd N{:}\,A}{\Gamma \rhd MN{:}\,B}\;(\supset_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M{:}\,A}{\Gamma \rhd \mathsf{inl}(M){:}\,A \vee B}\;(\vee_{\mathcal{I}-1}) \qquad \frac{\Gamma \rhd M{:}\,B}{\Gamma \rhd \mathsf{inr}(M){:}\,A \vee B}\;(\vee_{\mathcal{I}-2})$$

$$\frac{\Gamma \rhd M{:}\,A \vee B \qquad \Gamma, x{:}\,A \rhd N{:}\,C \qquad \Gamma, y{:}\,B \rhd P{:}\,C}{\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P{:}\,C}\;(\vee_{\mathcal{E}})$$

Figure 3.15: The Extended λ-Calculus

Although this term is in β-normal form, it is clearly *not* in c-normal form (the c-redex is underlined). This essentially reveals an open problem concerning the desired evaluation strategy of a linear functional language. We shall discuss this further in Chapter 5

As given in Definition 12 of Chapter 2, there is an alternative translation, $(-)^{\star}$, which for the $(\supset, \wedge)$-fragment of **IL** only involves the multiplicatives. Extending it to a translation on terms gives a translation which is the same as that in Figure 3.16 expect for the translation of pairs, which is as follows.

$$\begin{aligned}
|\langle M, N\rangle|^{\star}_{\Gamma} &\stackrel{\text{def}}{=} \mathsf{copy}\ \vec{x}\ \mathsf{as}\ \vec{y}, \vec{z}\ \mathsf{in} \\
&\quad\ (\mathsf{promote}\ \vec{y}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ |M|^{\star}_{\Gamma}) \otimes (\mathsf{promote}\ \vec{z}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ |N|^{\star}_{\Gamma}) \\
|\mathsf{fst}(M)|^{\star}_{\Gamma} &\stackrel{\text{def}}{=} \mathsf{let}\ |M|^{\star}_{\Gamma}\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ (\mathsf{discard}\ z\ \mathsf{in}\ \mathsf{derelict}(y)) \\
|\mathsf{snd}(M)|^{\star}_{\Gamma} &\stackrel{\text{def}}{=} \mathsf{let}\ |M|^{\star}_{\Gamma}\ \mathsf{be}\ y \otimes z\ \mathsf{in}\ (\mathsf{discard}\ y\ \mathsf{in}\ \mathsf{derelict}(z))
\end{aligned}$$

Of course, this translation function has the same desired property as the Girard translation.

Theorem 21. If $\vdash_{IL} \vec{x}{:}\,\Gamma \rhd M{:}\,A$ then $\vdash_{ILL} \vec{x}{:}\,!(\Gamma^{\star}) \rhd |M|^{\star}_{\Gamma}{:}\,A^{\star}$.

Proof. By structural induction. ■

Also in Chapter 2 we gave a translation, $(-)^{\mathsf{S}}$, from **ILL** to **IL**. Again we can apply the Curry-Howard correspondence to this translation to derive a translation from the linear term calculus to the extended λ-calculus. We give the translation procedure in Figure 3.17.

Of course, this translation procedure has the expected following property.

Theorem 22. If $\vdash_{ILL} \Gamma \rhd M{:}\,A$ then $\vdash_{IL} \Gamma \rhd |M|^{\mathsf{S}}{:}\,A^{\mathsf{S}}$.

Proof. By structural induction. ■

$$
\begin{array}{rcl}
|y|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{discard}\ \vec{x}\ \mathsf{in}\ \mathsf{derelict}(y) \\
|\lambda y{:}\, A.M|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \lambda y{:}\, !A^{\circ}.(|M|^{\circ}_{\Gamma\cup\{y\}}) \\
|MN|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{copy}\ \vec{x}\ \mathsf{as}\ \vec{x'}, \vec{x''} \\
& & \mathsf{in}\ ((|M[\vec{x} := \vec{x'}]|^{\circ}_{\Gamma'})(\mathsf{promote}\ \vec{x''}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ |N|^{\circ}_{\Gamma})) \\
|\langle M, N\rangle|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \langle |M|^{\circ}_{\Gamma}, |N|^{\circ}_{\Gamma}\rangle \\
|\mathsf{fst}(M)|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{fst}(|M|^{\circ}_{\Gamma}) \\
|\mathsf{snd}(M)|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{snd}(|M|^{\circ}_{\Gamma}) \\
|\nabla_A(M)|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{false}_A(\_\,;\ |M|^{\circ}_{\Gamma}) \\
|\mathsf{inl}(M)|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{inl}(\mathsf{promote}\ \vec{x}\ \mathsf{for}\ \vec{x'}\ \mathsf{in}\ (|M[\vec{x} := \vec{x'}]|^{\circ}_{\Gamma})) \\
|\mathsf{inr}(M)|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{inr}(\mathsf{promote}\ \vec{x}\ \mathsf{for}\ \vec{x'}\ \mathsf{in}\ (|M[\vec{x} := \vec{x'}]|^{\circ}_{\Gamma})) \\
\left| \begin{array}{ll} \mathsf{case}\ M\ \mathsf{of} & \mathsf{inl}(y) \to N \parallel \\ & \mathsf{inr}(z) \to P \end{array} \right|^{\circ}_{\Gamma} & \stackrel{\text{def}}{=} & \mathsf{copy}\ \vec{x}\ \mathsf{as}\ \vec{x'}, \vec{x''} \\
& & \mathsf{in}\ \mathsf{case}\ |(M[\vec{x} := \vec{x'}])|^{\circ}_{\Gamma'}\ \mathsf{of} \\
& & \quad \mathsf{inl}(y) \to |(N[\vec{x} := \vec{x''}])|^{\circ}_{\Gamma''\cup\{y\}} \\
& & \quad \mathsf{inr}(z) \to |(P[\vec{x} := \vec{x''}])|^{\circ}_{\Gamma''\cup\{z\}}
\end{array}
$$

where $\Gamma = \vec{x}$.

Figure 3.16: Translating λ-Terms into Linear Terms

$$
\begin{array}{rcl}
|x|^{\mathsf{S}} & \stackrel{\text{def}}{=} & x \\
|\lambda x{:}\, A.M|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \lambda x{:}\, A^{\mathsf{S}}.|M|^{\mathsf{S}} \\
|MN|^{\mathsf{S}} & \stackrel{\text{def}}{=} & |M|^{\mathsf{S}}|N|^{\mathsf{S}} \\
|M \otimes N|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \langle |M|^{\mathsf{S}}, |N|^{\mathsf{S}}\rangle \\
|\mathsf{let}\ M\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ N|^{\mathsf{S}} & \stackrel{\text{def}}{=} & |N|^{\mathsf{S}}[x := \mathsf{fst}(|M|^{\mathsf{S}}), y := \mathsf{snd}(|N|^{\mathsf{S}})] \\
|\langle M, N\rangle|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \langle |M|^{\mathsf{S}}, |N|^{\mathsf{S}}\rangle \\
|\mathsf{fst}(M)|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \mathsf{fst}(|M|^{\mathsf{S}}) \\
|\mathsf{snd}(M)|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \mathsf{snd}(|M|^{\mathsf{S}}) \\
|\mathsf{inl}(M)|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \mathsf{inl}(|M|^{\mathsf{S}}) \\
|\mathsf{inr}(M)|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \mathsf{inr}(|M|^{\mathsf{S}}) \\
\left| \begin{array}{ll} \mathsf{case}\ M\ \mathsf{of} & \mathsf{inl}(y) \to N \parallel \\ & \mathsf{inr}(z) \to P \end{array} \right|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \begin{array}{ll} \mathsf{case}\ |M|^{\mathsf{S}}\ \mathsf{of} & \mathsf{inl}(y) \to |N|^{\mathsf{S}} \parallel \\ & \mathsf{inr}(z) \to |P|^{\mathsf{S}} \end{array} \\
|\mathsf{false}_A(\vec{N}\,;\ M)|^{\mathsf{S}} & \stackrel{\text{def}}{=} & \nabla_A(|M|^{\mathsf{S}}) \\
|\mathsf{discard}\ M\ \mathsf{in}\ N|^{\mathsf{S}} & \stackrel{\text{def}}{=} & |N|^{\mathsf{S}} \\
|\mathsf{copy}\ M\ \mathsf{as}\ x, y\ \mathsf{in}\ N|^{\mathsf{S}} & \stackrel{\text{def}}{=} & |N|^{\mathsf{S}}[x, y := |M|^{\mathsf{S}}] \\
|\mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N|^{\mathsf{S}} & \stackrel{\text{def}}{=} & |N|^{\mathsf{S}}[\vec{x} := |\vec{M}|^{\mathsf{S}}]
\end{array}
$$

Figure 3.17: Translating Linear Terms into λ-Terms

Chapter 4
Categorical Analysis

1 Linear Equational Logic

Traditional (categorical) treatments of many-sorted equational logics [60, 22] assume that the rules for term formation automatically contain the structural rules of *Weakening*, *Contraction* and *Exchange*. This leads to the result that a categorical model for a many-sorted equational logic is at least a category with finite products. Here we are considering linear logic, where we do not have *Weakening* and *Contraction*, so we need to linearize the equational logic. We shall proceed in detail through the standard definitions, although, of course, in the linear set-up.

1.1 Signatures, Theories and Judgements

A *signature*, *Sg*, for a linear equational logic is specified by the following.

- A collection of *types*: σ, τ, ...
- A collection of *function symbols*: f, g, ...
- A *sorting* for each function symbol, which is a non empty list of types. We shall denote the sorting of f as $f\colon\sigma_1,\ldots,\sigma_n \to \tau$. If $n = 0$ then we say that f is a *constant* of type τ.

Suppose we are given a signature *Sg* for a linear equational logic, we can then define an *abstract syntax signature*, Σ, by two sets

1. $Ar = \{\text{TERM}\}$
2. $Con = \{f \in Sg \mid \text{if } f\colon\sigma_1,\ldots,\sigma_n \to \tau \text{ then } f\colon \text{TERM}^n \to \text{TERM}\} \cup \{v_1,\ldots\}$[1].

We say that the *raw terms* of a linear equational logic generated by the signature *Sg* are exactly the closed expressions of the abstract syntax generated by Σ which have arity TERM.

A *context* is a finite list of pairs of (object level) variables and types and is denoted

$$\Gamma = [x_1\colon\sigma_1,\ldots,x_n\colon\sigma_n].$$

It is assumed that the x_i are distinct. Given two disjoint contexts Γ and Δ we denote their concatenation by Γ,Δ and given a context Γ and a pair $y\colon\tau$, their concatenation by $\Gamma, y\colon\tau$.

A *term in context* consists of a raw term M and a context Γ containing all the variables occurring in M. We denote it using a judgement of the form

$$\Gamma \triangleright M\colon\sigma.$$

These judgements are generated by the rules

$$\frac{}{x\colon\sigma \triangleright x\colon\sigma}\ \textit{Identity},$$

and

$$\frac{\Gamma_1 \triangleright M_1\colon\sigma_1 \quad \ldots \quad \Gamma_n \triangleright M_n\colon\sigma_n \qquad \text{where } f\colon\sigma_1,\ldots,\sigma_n \to \tau \text{ is in } Sg}{\Gamma_1,\ldots,\Gamma_n \triangleright f(M_1,\ldots,M_n)\colon\tau}\ \textit{Sort}.$$

We can show that there are two further admissible rules.

[1] The v_i are object level variables.

Lemma 12.

1. If $\Delta \triangleright M{:}\,\sigma$ and $\Gamma, x{:}\,\sigma \triangleright N{:}\,\tau$ then $\Gamma, \Delta \triangleright N[x := M]{:}\,\tau$.
2. If $\Gamma, x{:}\,\sigma, y{:}\,\tau \triangleright M{:}\,\upsilon$ then $\Gamma, y{:}\,\tau, x{:}\,\sigma \triangleright M{:}\,\upsilon$.

Proof. Both by structural induction. ■

Writing these admissible rules out in full amounts to the rules

$$\frac{\Delta \triangleright M{:}\,\sigma \qquad \Gamma, x{:}\,\sigma \triangleright N{:}\,\tau}{\Gamma, \Delta \triangleright N[x := M]{:}\,\tau}\ \textit{Substitution},$$

and

$$\frac{\Gamma, x{:}\,\sigma, y{:}\,\tau \triangleright M{:}\,\upsilon}{\Gamma, y{:}\,\tau, x{:}\,\sigma \triangleright M{:}\,\upsilon}\ \textit{Exchange}.$$

An *algebraic theory*, *Th*, is a pair (*Sg*, *Ax*), where *Sg* is a signature and *Ax* is a collection of equations in context. An *equation in context* is of the form

$$\Gamma \triangleright M = N{:}\,\sigma,$$

where $\Gamma \triangleright M{:}\,\sigma$ and $\Gamma \triangleright N{:}\,\sigma$ are (well formed) terms in context.

The equations in *Ax* are often referred to as the *axioms* of the theory *Th*. The *theorems* of *Th* consists of the least collection of equations in context which contain the axioms of *Th* and are closed under the rules

$$\frac{\Gamma \triangleright M{:}\,\sigma}{\Gamma \triangleright M = M{:}\,\sigma}\ \textit{Reflexive}$$

$$\frac{\Gamma \triangleright M = N{:}\,\sigma}{\Gamma \triangleright N = M{:}\,\sigma}\ \textit{Symmetry}$$

$$\frac{\Gamma \triangleright M = N{:}\,\sigma \qquad \Gamma \triangleright N = P{:}\,\sigma}{\Gamma \triangleright M = P{:}\,\sigma}\ \textit{Transitive}$$

$$\frac{\Delta \triangleright M = N{:}\,\sigma \qquad \Gamma, x{:}\,\sigma \triangleright P = Q{:}\,\tau}{\Gamma, \Delta \triangleright P[x := M] = Q[x := N]{:}\,\tau}\ \textit{Substitution}.$$

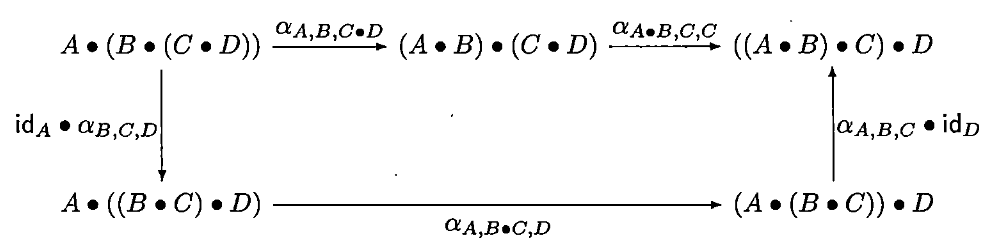
2 Categorical Semantics for Linear Equational Logic

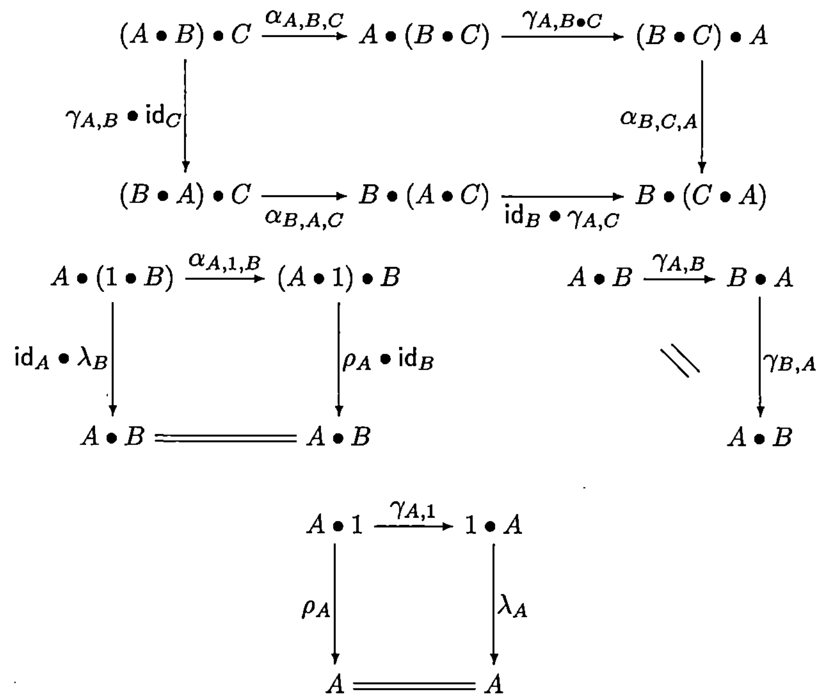
For completeness let us first define a symmetric monoidal category.

Definition 22. A *symmetric monoidal category (SMC)*, $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$, is a category $\mathbb{C}$ equipped with a bifunctor $\bullet{:}\,\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with a neutral element 1 and natural isomorphisms α, λ, ρ and γ:

1. $\alpha_{A,B,C}{:}\,A \bullet (B \bullet C) \xrightarrow{\sim} (A \bullet B) \bullet C$
2. $\lambda_A{:}\,1 \bullet A \xrightarrow{\sim} A$
3. $\rho_A{:}\,A \bullet 1 \xrightarrow{\sim} A$
4. $\gamma_{A,B}{:}\,A \bullet B \xrightarrow{\sim} B \bullet A$,

which make the following 'coherence' diagrams commute.





The following equality is also required to hold:

$$\lambda_1 \quad = \quad \rho_1 : 1 \bullet 1 \to 1$$

Definition 23. A *symmetric monoidal closed category (SMCC)*, $(\mathbb{C}, \bullet, \multimap, 1, \alpha, \lambda, \rho, \gamma)$, is a SMC such that for all objects A in $\mathbb{C}$, the functor $-\otimes A$ has a specified right adjoint $A \multimap -$.

Let $\mathbb{C}$ be a SMC $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$. A *structure*, $\mathcal{M}$, in $\mathbb{C}$ for a given signature Sg is specified by giving an object $[\![\sigma]\!]$ in $\mathbb{C}$ for each type σ, and a morphism $[\![f]\!]: [\![\sigma_1]\!] \bullet \ldots \bullet [\![\sigma_n]\!] \to [\![\tau]\!]$ in $\mathbb{C}$ for each function symbol $f: \sigma_1, \ldots, \sigma_n \to \tau$. In the case where $n = 0$ then the structure assigns a morphism $[\![c]\!]: 1 \to [\![\tau]\!]$ to a constant $c: \tau$.

Given a context $\Gamma = [x_1: \sigma_1, \ldots, x_n: \sigma_n]$ we define $[\![\Gamma]\!]$ to be the product $[\![\sigma_1]\!] \bullet \ldots \bullet [\![\sigma_n]\!]$. We represent the empty context with the neutral element 1. We need to define the bracketing convention. It shall be assumed that the tensor product is *left associative*, i.e. $A_1 \bullet A_2 \bullet \ldots \bullet A_n$ will be taken to mean $(\ldots(A_1 \bullet A_2) \bullet \ldots) \bullet A_n$. We find it useful to define two 'book-keeping' functions,

$$\mathsf{Split}(\Gamma, \Delta): [\![\Gamma, \Delta]\!] \to [\![\Gamma]\!] \bullet [\![\Delta]\!]$$

$$\mathsf{Split}(\Gamma, \Delta) \stackrel{\text{def}}{=} \begin{cases} \lambda_\Delta^{-1} & \text{If } \Gamma \text{ empty} \\ \rho_\Gamma^{-1} & \text{If } \Delta \text{ empty} \\ \mathsf{id}_{\Gamma \bullet A} & \text{If } \Delta = A \\ \mathsf{Split}(\Gamma, \Delta') \bullet \mathsf{id}_A ; \alpha_{\Gamma,\Delta',A}^{-1} & \text{If } \Delta = \Delta', A \end{cases}$$

$$\mathsf{Join}(\Gamma, \Delta): [\![\Gamma, \Delta]\!] \to [\![\Gamma]\!] \bullet [\![\Delta]\!]$$

$$\mathsf{Join}(\Gamma, \Delta) \stackrel{\text{def}}{=} \begin{cases} \lambda_\Delta & \text{If } \Gamma \text{ empty} \\ \rho_\Gamma & \text{If } \Delta \text{ empty} \\ \mathsf{id}_{\Gamma \bullet A} & \text{If } \Delta = A \\ \alpha_{\Gamma,\Delta',A} ; \mathsf{Join}(\Gamma, \Delta') \bullet \mathsf{id}_A & \text{If } \Delta = \Delta', A \end{cases}$$

We shall also refer to indexed variants of these; for example

$$\mathsf{Split_n}(\Gamma_1, \ldots, \Gamma_n): [\![\Gamma_1, \ldots, \Gamma_n]\!] \to [\![\Gamma_1]\!] \bullet \ldots \bullet [\![\Gamma_n]\!],$$

which is defined in the obvious way.

The semantics of a term in context is then specified by a structural induction on the term.

$$[\![x:\sigma \rhd x:\sigma]\!] \stackrel{\text{def}}{=} \mathsf{id}_\sigma$$

$$[\![\Gamma_1,\ldots,\Gamma_n \rhd f(M_1,\ldots,M_n):\tau]\!] \stackrel{\text{def}}{=} \mathsf{Split_n}(\Gamma_1,\ldots,\Gamma_n);[\![\Gamma_1 \rhd M_1:\sigma_1]\!] \bullet \ldots \bullet [\![\Gamma_n \rhd M_n:\sigma_n]\!];[\![f]\!]$$

We have seen earlier that we can consider two further admissible rules for forming terms in context. Let us consider the categorical import of these rules; firstly the *Exchange* rule.

Lemma 13. Let $\Gamma, x:\sigma, y:\tau \rhd M:v$ be a valid term in context, then

$$[\![\Gamma, y:\tau, x:\sigma \rhd M:v]\!] = \alpha^{-1}_{\Gamma,\tau,\sigma};\mathsf{id}_\Gamma \bullet \gamma_{\tau,\sigma};\alpha_{\Gamma,\sigma,\tau};[\![\Gamma, x:\sigma, y:\tau \rhd M:v]\!].$$

This essentially means that the *Exchange* rule is handled implicitly by the symmetry of the model (i.e. the γ natural isomorphism). Now let us consider the *Substitution* rule.

Lemma 14. Let $x_1:\sigma_1,\ldots,x_n:\sigma_n \rhd M:\tau$, $\Gamma_1 \rhd M_1:\sigma_1$, $\ldots$, $\Gamma_n \rhd M_n:\sigma_n$ be valid terms in context then

$$\begin{array}{l}[\![\Gamma_1,\ldots,\Gamma_n \rhd M[x_1 := N_1,\ldots,x_n := N_n]:\tau]\!] = \\ \quad \mathsf{Split_n}(\Gamma_1,\ldots,\Gamma_n);[\![\Gamma_1 \rhd N_1:\sigma_1]\!] \bullet \ldots \bullet [\![\Gamma_n \rhd N_n:\sigma_n]\!];[\![x_1:\sigma_1,\ldots,x_n:\sigma_n \rhd M:\tau]\!].\end{array}$$

Proof. By induction on the derivation. ■

The above lemma represents an important feature of categorical logic and one which will be central to this chapter and as such deserves to be repeated: *substitution in the term calculus corresponds to composition in the category.*

Let $\mathcal{M}$ be a structure for a signature *Sg* in a SMC $\mathbb{C}$. Given an equation in context for the signature

$$\Gamma \rhd M = N:\sigma,$$

we say that the structure *satisfies* the equation if the morphisms it assigns to $\Gamma \rhd M:\sigma$ and $\Gamma \rhd N:\sigma$ are equal. Then given an algebraic theory $Th = (Sg, Ax)$, if a structure $\mathcal{M}$ satisfies all the equations in *Ax* it is called a *model.*

Theorem 23. Let $\mathbb{C}$ be a SMC, *Th* an algebraic theory and $\mathcal{M}$ a model of *Th* in $\mathbb{C}$. Then $\mathcal{M}$ satisfies any equation in context which is a theorem of *Th.*

Proof. Essentially we are trying to show that if $\Gamma \rhd M = N:\sigma$ is a theorem then $[\![\Gamma \rhd M:\sigma]\!] = [\![\Gamma \rhd N:\sigma]\!]$. Since $\mathcal{M}$ is a model we know that it satisfies all the axioms of *Th.* Thus we need to show that it satisfies all the equations given in §1.1. It is trivial to see that it satisfies *Reflexive, Symmetric* and *Transitive.* It satisfies *Substitution* by induction and from Lemma 14. ■

3 Analysing the Linear Term Calculus

In the previous section we have seen that to model an equational logic with just the structural rules of *Identity, Exchange* and *Substitution,* we need a SMC. We take this as a basis and consider the linear term calculus from Chapter 3. In that chapter we identified a series of β-reduction rules. Here we shall take those as *equalities* to form equations in contexts.

After giving the signature for the linear term calculus we shall consider each rule in turn to discover what extra structure is needed on top of a SMC to model the calculus. We shall also make some simple assumptions which will uncover some η-like rules as well as some rules which follow from considerations of naturality.

3.1 Preliminaries

Definition 24. A linear term calculus signature (LTC-signature), $\mathcal{L}$, is given by

- a collection of *types*. We have a collection of ground types which contains the distinguished ground types I, $\mathbf{t}$ and $\mathbf{f}$. The collection of types is given by the following grammar

$$A \stackrel{\text{def}}{=} \gamma \mid A \otimes A \mid A \multimap A \mid A \& A \mid A \oplus A \mid !A$$

  where γ represents any ground type,
- a collection of *function symbols*, containing the distinguished symbols **App**, $\mathbf{Lam}_A$, **Tensor**, $\mathbf{Split}_{A_1,A_2}$, **Unit**, **Let**, **With**, **Fst**, **Snd**, **Inl**, **Inr**, **Case**, **True**, $\mathbf{False}_A$, $\mathbf{Copy}_A$, $\mathbf{Discard}_A$, **Derelict** and $\mathbf{Promote}_{A_1,\ldots,A_n}$, and
- a *sorting* for each function symbol.

Given such a LTC-signature, we can define an abstract syntax signature $\Sigma = (GAr, Con)$. The collection of ground arities is just the set $\{\text{TERM}\}$. The collection of constants Con consists of the basic function symbols, which are given the arity $\text{TERM}^n \to \text{TERM}$ whenever they have the sorting $A_1, \ldots, A_n \to A_{n+1}$ in Sg, a countably infinite set of object level variables with arity TERM, together with the distinguished function symbols. To these distinguished function symbols we assign the following arities:

- **App**: TERM→ TERM→ TERM
- $\mathbf{Lam}_A$: (TERM→ TERM) → TERM
- **Tensor**: TERM→ TERM→ TERM
- **Split**: TERM→ (TERM^2 → TERM)→ TERM
- **Unit**: TERM
- **Let**: TERM→ TERM→ TERM
- **With**: TERM→ TERM→ TERM
- **Fst,Snd**: TERM→ TERM
- **Inl,Inr**: TERM→ TERM
- **Case**: TERM→ (TERM→ TERM) → (TERM→ TERM) → TERM
- **True**: TERM^n → TERM
- **False**:TERM→(TERM^n →TERM)
- **Copy**: TERM→ (TERM^2 → TERM)→ TERM
- **Discard**: TERM→ TERM→ TERM
- **Derelict**: TERM→ TERM
- **Promote**: TERM^n →(TERM^n →TERM)→ TERM

We can now give the rules for forming (object level) terms in context. These judgements are the same as for the simple linear equational logic but from the enriched signature $\mathcal{L}$ given in Definition 24. These judgements are as follows.

$$\frac{}{x\colon A \rhd x\colon A}\ \textit{Identity}$$

$$\frac{\Gamma_1 \rhd M_1\colon A_1 \quad \ldots \quad \Gamma_n \rhd M_n\colon A_n \qquad \text{where } f\colon A_1,\ldots,A_n \to B \text{ is in } Sg}{\Gamma_1,\ldots,\Gamma_n \rhd f(M_1,\ldots,M_n)\colon B}\ \textit{Sort}$$

$$\frac{\Gamma, x\colon A \rhd M\colon B}{\Gamma \rhd \mathbf{Lam}_A((x)M)\colon A{-\!\circ}B}\ ({-\!\circ}_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon A{-\!\circ}B \qquad \Delta \rhd N\colon A}{\Gamma,\Delta \rhd \mathbf{App}(M,N)\colon B}\ ({-\!\circ}_{\mathcal{E}})$$

$$\frac{}{\rhd \mathbf{Unit}\colon I}\ (I_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon I}{\Gamma,\Delta \rhd \mathbf{Let}(M,N)\colon A}\ (I_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon B}{\Gamma,\Delta \rhd \mathbf{Tensor}(M,N)\colon A{\otimes}B}\ (\otimes_{\mathcal{I}}) \qquad \frac{\Delta \rhd M\colon A{\otimes}B \qquad \Gamma, x\colon A, y\colon B \rhd N\colon C}{\Gamma,\Delta \rhd \mathbf{Split}(M,(x,y)N)\colon C}\ (\otimes_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon B}{\Gamma,\Delta \rhd \mathbf{With}(M,N)\colon A\&B}\ (\&_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon A\&B}{\Gamma \rhd \mathbf{Fst}(M)\colon A}\ (\&_{\mathcal{E}-1}) \qquad \frac{\Gamma \rhd M\colon A\&B}{\Gamma \rhd \mathbf{Snd}(M)\colon B}\ (\&_{\mathcal{E}-2})$$

$$\frac{\Gamma_1 \rhd M_1\colon A_1 \ \cdots\ \Gamma_n \rhd M_n\colon A_n}{\Gamma_1,\ldots,\Gamma_n \rhd \mathbf{True}(\vec{M})\colon \mathbf{t}}\ (\mathbf{t}_{\mathcal{I}}) \qquad \frac{\Gamma_1 \rhd M_1\colon A_1 \ \cdots\ \Gamma_n \rhd M_n\colon A_n \qquad \Delta \rhd N\colon \mathbf{f}}{\Gamma_1,\ldots,\Gamma_n,\Delta \rhd \mathbf{False}_B(\vec{M},N)\colon B}\ (\mathbf{f}_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A}{\Gamma \rhd \mathbf{Inl}(M)\colon A \oplus B}\ (\oplus_{\mathcal{I}-1}) \qquad \frac{\Gamma \rhd M\colon B}{\Gamma \rhd \mathbf{Inr}(M)\colon A \oplus B}\ (\oplus_{\mathcal{I}-2})$$

$$\frac{\Gamma \rhd M\colon A \oplus B \qquad \Delta, x\colon A \rhd N\colon C \qquad \Delta, y\colon B \rhd P\colon C}{\Gamma,\Delta \rhd \mathbf{Case}(M,(x)N,(y)P)\colon C}\ (\oplus_{\mathcal{E}})$$

$$\frac{\Delta_1 \rhd M_1\colon !A_1 \ \cdots\ \Delta_n \rhd M_n\colon !A_n \qquad x_1\colon !A_1,\ldots,x_n\colon !A_n \rhd N\colon B}{\Delta_1,\ldots,\Delta_n \rhd \mathbf{Promote}(M_1,\ldots,M_n,(x_1,\ldots,x_n)N)\colon !B}\ \textit{Promotion}$$

$$\frac{\Gamma \rhd M\colon !A \qquad \Delta \rhd N\colon B}{\Gamma,\Delta \rhd \mathbf{Discard}(M,N)\colon B}\ \textit{Weakening}$$

$$\frac{\Delta \rhd M\colon !A \qquad \Gamma, x\colon !A, y\colon !A \rhd N\colon B}{\Gamma,\Delta \rhd \mathbf{Copy}(M,(x,y)N)\colon B}\ \textit{Contraction}$$

$$\frac{\Gamma \rhd M\colon !A}{\Gamma \rhd \mathbf{Derelict}(M)\colon A}\ \textit{Dereliction}$$

However to keep consistency with the rest of this thesis we shall not use this Martin-Löf-style notation for the linear term calculus. Rather we shall use the same syntax as in Chapter 3. Of course, the two notations are equivalent and before proceeding we shall list this equivalence.

$$\begin{array}{rcl}
\mathbf{App}(M,N) & \Longleftrightarrow & MN \\
\mathbf{Lam}_A((x)M) & \Longleftrightarrow & \lambda x\colon A.M \\
\mathbf{Tensor}(M,N) & \Longleftrightarrow & M{\otimes}N \\
\mathbf{Split}(M,(x,y)N) & \Longleftrightarrow & \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N \\
\mathbf{Unit} & \Longleftrightarrow & * \\
\mathbf{Let}(M,N) & \Longleftrightarrow & \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N
\end{array}$$

$$\begin{array}{rcl}
\mathbf{With}(M,N) & \Longleftrightarrow & \langle M,N\rangle \\
\mathbf{Fst}(M) & \Longleftrightarrow & \mathsf{fst}(M) \\
\mathbf{Snd}(M) & \Longleftrightarrow & \mathsf{snd}(N) \\
\mathbf{Inl}(M) & \Longleftrightarrow & \mathsf{inl}(M) \\
\mathbf{Inr}(M) & \Longleftrightarrow & \mathsf{inr}(M) \\
\mathbf{Case}(M,(x)N,(y)P) & \Longleftrightarrow & \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \rightarrow N \parallel \mathsf{inr}(y) \rightarrow P \\
\mathbf{True}(\vec{M}) & \Longleftrightarrow & \mathsf{true}(\vec{M}) \\
\mathbf{False}_A(\vec{M},N) & \Longleftrightarrow & \mathsf{false}_A(\vec{M}\ ;\ N) \\
\mathbf{Copy}(M,(x,y)N) & \Longleftrightarrow & \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N \\
\mathbf{Discard}(M,N) & \Longleftrightarrow & \mathsf{discard}\ M\ \mathsf{in}\ N \\
\mathbf{Promote}(\vec{M},(\vec{x})N) & \Longleftrightarrow & \mathsf{promote}\ \vec{M}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ N
\end{array}$$

Using this more familiar syntax the judgements for forming valid terms in context are repeated in Figure 4.1.

It is easy to check that within this enriched theory the rules for *Substitution* and *Exchange* are still admissible.

A linear term calculus theory (LTC-theory), $\mathcal{T}$, is a pair $(\mathcal{L},\mathcal{A})$ where $\mathcal{L}$ is a LTC-signature and $\mathcal{A}$ is a collection of equations in context. An equation in context, as before, takes the form

$$\Gamma \triangleright M = N : A,$$

where $\Gamma \triangleright M : A$ and $\Gamma \triangleright N : A$ are valid terms in context as generated by the rules in Figure 4.1. The equations in $\mathcal{A}$ are known as *axioms* of the theory. To start we shall take the β-rules from Chapter 3 and consider them as equations in context, although later we shall include some extra equations. The *theorems* of $\mathcal{T}$ are then the least collection of equations in context which contain the axioms $\mathcal{A}$ and are closed under the β-equations and those from §1.1. These rules are given in Figure 4.2.[2]

3.2 Analysis

Starting with a SMC $(\mathbb{C},\bullet,1,\alpha,\lambda,\rho,\gamma)$, we shall take each connective in turn and consider its categorical import. We shall see that the category theory employed will also suggest some new equations in context.

Linear Implication

The introduction rule for linear implication is of the form

$$\frac{\Gamma, x{:}\,A \triangleright M : B}{\Gamma \triangleright \lambda x{:}\,A.M : A \multimap B}\ (\multimap_{\mathcal{I}}).$$

We know that a term in context is represented as a morphism. A logical rule is then modelled by an operation on morphisms, or, in other words, on Hom-sets. The introduction rule above suggests an operation on the Hom-sets of the form

$$\Phi : \mathbb{C}(\Gamma \bullet A, B) \rightarrow \mathbb{C}(\Gamma, A \multimap B).$$

We know that in terms of the syntax we can substitute freely for any of the free variables of $\lambda x{:}\,A.M$, i.e. those contained in Γ. Since substitution in the term calculus is modelled by composition in the

[2] As is standard, we shall assume the obvious *congruence* equations in context; for example

$$\frac{\Gamma, x{:}\,A \triangleright M = N : B}{\Gamma \triangleright \lambda x{:}\,A.M = \lambda x{:}\,A.N : A \multimap B}\ {\multimap}\ \mathit{Cong}$$

$$\frac{}{x\colon A \rhd x\colon A}\ \textit{Identity}$$

$$\frac{\Gamma_1 \rhd M_1\colon A_1 \quad \ldots \quad \Gamma_n \rhd M_n\colon A_n \qquad \text{where } f\colon A_1,\ldots,A_n \to B \text{ is in } Sg}{\Gamma_1,\ldots,\Gamma_n \rhd f(M_1,\ldots,M_n)\colon B}\ \textit{Sort}$$

$$\frac{\Gamma, x\colon A \rhd M\colon B}{\Gamma \rhd \lambda x\colon A.M\colon A\multimap B}\ (\multimap_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon A\multimap B \qquad \Delta \rhd N\colon A}{\Gamma,\Delta \rhd MN\colon B}\ (\multimap_{\mathcal{E}})$$

$$\frac{}{\rhd *\colon I}\ (I_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon I}{\Gamma,\Delta \rhd \mathsf{let}\ N\ \mathsf{be}\ *\ \mathsf{in}\ N\colon A}\ (I_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon B}{\Gamma,\Delta \rhd M\otimes N\colon A\otimes B}\ (\otimes_{\mathcal{I}}) \qquad \frac{\Delta \rhd M\colon A\otimes B \qquad \Gamma, x\colon A, y\colon B \rhd N\colon C}{\Gamma,\Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ N\colon C}\ (\otimes_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon B}{\Gamma,\Delta \rhd (M,N)\colon A\& B}\ (\&_{\mathcal{I}}) \qquad \frac{\Gamma \rhd M\colon A\& B}{\Gamma \rhd \mathsf{fst}(M)\colon A}\ (\&_{\mathcal{E}-1}) \qquad \frac{\Gamma \rhd M\colon A\& B}{\Gamma \rhd \mathsf{snd}(M)\colon B}\ (\&_{\mathcal{E}-2})$$

$$\frac{\Gamma_1 \rhd M_1\colon A_1 \quad \cdots \quad \Gamma_n \rhd M_n\colon A_1}{\Gamma_1,\ldots,\Gamma_n \rhd \mathsf{true}(\vec{M})\colon \mathbf{t}}\ (\mathbf{t}_{\mathcal{I}}) \qquad \frac{\Gamma_1 \rhd M_1\colon A_1 \quad \cdots \quad \Gamma_n \rhd M_n\colon A_n \qquad \Delta \rhd N\colon \mathbf{f}}{\Gamma_1,\ldots,\Gamma_n,\Delta \rhd \mathsf{false}_B(\vec{M}\ ;\ N)\colon B}\ (\mathbf{f}_{\mathcal{E}})$$

$$\frac{\Gamma \rhd M\colon A}{\Gamma \rhd \mathsf{inl}(M)\colon A\oplus B}\ (\oplus_{\mathcal{I}-1}) \qquad \frac{\Gamma \rhd M\colon B}{\Gamma \rhd \mathsf{inr}(M)\colon A\oplus B}\ (\oplus_{\mathcal{I}-2})$$

$$\frac{\Delta \rhd M\colon A\oplus B \qquad \Gamma, x\colon A \rhd N\colon C \qquad \Gamma, y\colon B \rhd P\colon C}{\Gamma,\Delta \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P\colon C}\ (\oplus_{\mathcal{E}})$$

$$\frac{\Gamma_1 \rhd M_1\colon !A_1 \quad \cdots \quad \Gamma_n \rhd M_n\colon !A_n \qquad x_1\colon !A_1,\ldots,x_n\colon !A_n \rhd N\colon B}{\Gamma_1,\ldots,\Gamma_n \rhd \mathsf{promote}\ M_1,\ldots,M_n\ \mathsf{for}\ x_1,\ldots,x_n\ \mathsf{in}\ N\colon !B}\ \textit{Promotion}$$

$$\frac{\Gamma \rhd M\colon !A \qquad \Delta \rhd N\colon B}{\Gamma,\Delta \rhd \mathsf{discard}\ M\ \mathsf{in}\ N\colon B}\ \textit{Weakening}$$

$$\frac{\Delta \rhd M\colon !A \qquad \Gamma, x\colon !A, y\colon !A \rhd N\colon B}{\Gamma,\Delta \rhd \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N\colon B}\ \textit{Contraction}$$

$$\frac{\Gamma \rhd M\colon !A}{\Gamma \rhd \mathsf{derelict}(M)\colon A}\ \textit{Dereliction}$$

Figure 4.1: Term in Context Judgements for the Linear Term Calculus.

$$\frac{\Gamma \triangleright M : \sigma}{\Gamma \triangleright M = M : \sigma}\ \mathit{Reflexive} \qquad \frac{\Gamma \triangleright M = N : \sigma}{\Gamma \triangleright N = M : \sigma}\ \mathit{Symmetry}$$

$$\frac{\Gamma \triangleright M = N : \sigma \qquad \Gamma \triangleright N = P : \sigma}{\Gamma \triangleright M = P : \alpha}\ \mathit{Transitive}$$

$$\frac{\Delta \triangleright M = N : \sigma \qquad \Gamma, x : \sigma \triangleright P = Q : \tau}{\Gamma, \Delta \triangleright P[x := M] = Q[x := N] : \tau}\ \mathit{Substitution}$$

$$\frac{\Gamma, x : A \triangleright M : B \qquad \Delta \triangleright N : A}{\Gamma, \Delta \triangleright (\lambda x : A.M)N = M[x := N] : B}\ {\multimap}_{Eq}$$

$$\frac{\Gamma \triangleright M : A \qquad \triangleright * : I}{\Gamma \triangleright \mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ M = M : A}\ I_{Eq}$$

$$\frac{\Gamma \triangleright M : A \qquad \Delta \triangleright N : B \qquad \Theta, x : A, y : B \triangleright P : C}{\Gamma, \Delta, \Theta \triangleright \mathsf{let}\ M \otimes N\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ P = P[x := M, y := N] : C}\ \otimes_{Eq}$$

$$\frac{\Gamma \triangleright M : A \qquad \Gamma \triangleright N : B}{\Gamma \triangleright \mathsf{fst}(\langle M, N \rangle) = M : A}\ \&_{Eq-1} \qquad \frac{\Gamma \triangleright M : A \qquad \Gamma \triangleright N : B}{\Gamma \triangleright \mathsf{snd}(\langle M, N \rangle) = N : B}\ \&_{Eq-2}$$

$$\frac{\Delta \triangleright M : A \qquad \Gamma, x : A \triangleright N : C \qquad \Gamma, y : B \triangleright P : C}{\Gamma, \Delta \triangleright \mathsf{case\ inl}(M)\ \mathsf{of\ inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P = N[x := M] : C}\ \oplus_{Eq-1}$$

$$\frac{\Delta \triangleright M : A \qquad \Gamma, x : A \triangleright N : C \qquad \Gamma, y : B \triangleright P : C}{\Gamma, \Delta \triangleright \mathsf{case\ inr}(M)\ \mathsf{of\ inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P = P[y := M] : C}\ \oplus_{Eq-2}$$

$$\frac{\Gamma_1 \triangleright M_1 : !A_1 \quad \ldots \quad \Gamma_n \triangleright M_n : !A_n \qquad x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B}{\begin{array}{ll}\Gamma_1, \ldots, \Gamma_n \triangleright & \mathsf{derelict}(\mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N) \\ & = N[x_1 := M_1, \ldots, x_n := M_n] : B\end{array}}\ \mathit{Der}_{Eq}$$

$$\frac{\begin{array}{l}\Gamma_1 \triangleright M_1 : !A_1 \\ \Gamma_n \triangleright M_n : !A_n\end{array} \qquad x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B \qquad \Delta \triangleright P : C}{\begin{array}{ll}\Gamma_1, \ldots, \Gamma_n, \Delta \triangleright & \mathsf{discard}\ (\mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N)\ \mathsf{in}\ P \\ & = \mathsf{discard}\ M_1, \ldots, M_n\ \mathsf{in}\ P : C\end{array}}\ \mathit{Disc}_{Eq}$$

$$\frac{\begin{array}{l}\Gamma_1 \triangleright M_1 : !A_1 \\ \Gamma_n \triangleright M_n : !A_n\end{array} \qquad x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B \qquad \Delta, x : !B, y : !B \triangleright P : C}{\begin{array}{lll}\Gamma_1, \ldots, \Gamma_n, \Delta \triangleright & \multicolumn{2}{l}{\mathsf{copy}\ (\mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N)\ \mathsf{as}\ x, y\ \mathsf{in}\ P} \\ & = & \mathsf{copy}\ M_1, \ldots, M_n\ \mathsf{as}\ (x_1', \ldots, x_n'), (x_1'', \ldots, x_n'')\ \mathsf{in} \\ & & P[x := \mathsf{promote}\ x_1', \ldots, x_n'\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N, \\ & & \quad y := \mathsf{promote}\ x_1'', \ldots, x_n''\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N] : C\end{array}}\ \mathit{Copy}_{Eq}$$

Figure 4.2: Equations in Context for linear term calculus

category (Lemma 14), this means that Φ should be a natural transformation which is natural in Γ. Taking morphisms $m: \Gamma \bullet A \to B$ and $c: \Gamma' \to \Gamma$, then naturality gives the equation

$$c; \Phi_\Gamma(m) = \Phi_{\Gamma'}((c \bullet \mathsf{id}_A); m).$$

Let us write Cur instead of Φ. Then we make the definition

$$[\![\Gamma \rhd \lambda x: A.M: A \multimap B]\!] \stackrel{\text{def}}{=} \mathsf{Cur}([\![\Gamma, x: A \rhd M: B]\!]). \tag{4.1}$$

The elimination rule for linear implication is of the form

$$\frac{\Gamma \rhd M: A \multimap B \qquad \Delta \rhd N: A}{\Gamma, \Delta \rhd MN: B} (\multimap_{\mathcal{E}}),$$

which suggests a natural transformation with components

$$\Psi_{\Gamma,\Delta}: \mathbb{C}(\Gamma, A \multimap B) \times \mathbb{C}(\Delta, A) \to \mathbb{C}((\Gamma, \Delta), B).$$

Taking morphisms $e: \Gamma \to A \multimap B$, $f: \Delta \to A$, $c: \Gamma' \to \Gamma$, $d: \Delta' \to \Delta$, applying naturality gives the equation

$$\mathsf{Split}(\Gamma', \Delta'); c \bullet d; \mathsf{Join}(\Gamma, \Delta); \Psi_{\Gamma,\Delta}(e, f) \quad = \quad \Psi_{\Gamma',\Delta'}(c; e, d; f).$$

In particular taking e to be $\mathsf{id}_{A \multimap B}$, f to be id_A, c to be a morphism $m: \Gamma \to A \multimap B$ and d to be a morphism $n: \Delta \to A$, then by naturality

$$\mathsf{Split}(\Gamma, \Delta); m \bullet n; \Psi_{A \multimap B, A}(\mathsf{id}_{A \multimap B}, \mathsf{id}_A) \quad = \quad \Psi_{\Gamma,\Delta}(m, n).$$

Defining the morphism App $\stackrel{\text{def}}{=} \Psi_{A \multimap B, A}(\mathsf{id}_{A \multimap B}, \mathsf{id}_A)$ then we can make the definition

$$[\![\Gamma, \Delta \rhd MN: B]\!] \stackrel{\text{def}}{=} \mathsf{Split}(\Gamma, \Delta); [\![\Gamma \rhd M: A \multimap B]\!] \bullet [\![\Delta \rhd N: B]\!]; \mathsf{App}. \tag{4.2}$$

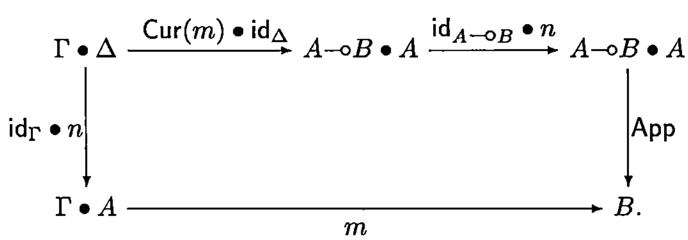
We have the following equation in context for the linear implication

$$\frac{\Gamma, x: A \rhd M: B \qquad \Delta \rhd N: A}{\Gamma, \Delta \rhd (\lambda x: A.M)N = M[x := N]: B} \multimap_{Eq}.$$

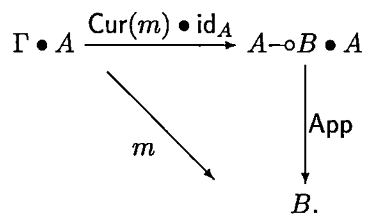
Taking morphisms $m: \Gamma \bullet A \to B$ and $n: \Delta \to A$, this rule amounts to the condition

$$\mathsf{Split}(\Gamma, \Delta); \mathsf{Cur}(m) \bullet n; \mathsf{App} \quad = \quad \mathsf{Split}(\Gamma, \Delta); \mathsf{id}_\Gamma \bullet n; m, \tag{4.3}$$

or diagrammatically



In particular, taking $\Delta \equiv A$ and $n = \mathsf{id}_A$ then



We shall make the simplifying assumption that this factorization is *unique*, namely that a given morphism $m: \Gamma \bullet A \to B$ can be factored uniquely as

$$\mathsf{Cur}(m) \bullet \mathsf{id}_\Gamma; \mathsf{App} \quad = \quad m. \tag{4.4}$$

Let us consider the case where $m = \mathsf{App}$. Then $\mathsf{Cur}(\mathsf{App}) \bullet \mathsf{id}; \mathsf{App} = \mathsf{App}$ and from the uniqueness assumption we can conclude that

$$\mathsf{Cur}(\mathsf{App}) \quad = \quad \mathsf{id}_{A \multimap B}. \tag{4.5}$$

We can define two natural transformations

$$\begin{array}{l} (-)^*: \mathbb{C}(\Gamma \bullet A, B) \to \mathbb{C}(\Gamma, A \multimap B) \\ f \mapsto \mathsf{Cur}(f), \end{array}$$

and

$$\begin{array}{l} (-)_*: \mathbb{C}(\Gamma, A \multimap B) \to \mathbb{C}(\Gamma \bullet A, B) \\ g \mapsto g \bullet \mathsf{id}_A; \mathsf{App}. \end{array}$$

Let us consider their composition. First take a morphism $f: \Gamma \bullet A \to B$, then

$$\begin{array}{rcll} ((f)^*)_* & = & (\mathsf{Cur}(f))^* & \\ & = & \mathsf{Cur}(f) \bullet \mathsf{id}_A; \mathsf{App} & \\ & = & f & \text{Rule 4.4.} \end{array}$$

Alternatively take a morphism $g: \Gamma \to A \multimap B$, then

$$\begin{array}{rcll} ((g)_*)^* & = & (g \bullet \mathsf{id}_A; \mathsf{App})^* & \\ & = & \mathsf{Cur}(g \bullet \mathsf{id}_A; \mathsf{App}) & \\ & = & g; \mathsf{Cur}(\mathsf{App}) & \text{Naturality of } \mathsf{Cur} \\ & = & g & \text{Rule 4.5.} \end{array}$$

Considering in detail this equality and taking a morphism $n: \Gamma \to A \multimap B$ gives

$$\begin{array}{rcll} n & = & \mathsf{Cur}(n \bullet \mathsf{id}_A; \mathsf{App}) & \\ [\![\Gamma \rhd N: A \multimap B]\!] & = & \mathsf{Cur}([\![\Gamma \rhd N: A \multimap B]\!] \bullet [\![x: A \rhd x: A]\!]; \mathsf{App}) & \\ [\![\Gamma \rhd N: A \multimap B]\!] & = & \mathsf{Cur}([\![\Gamma, x: A \rhd Nx: B]\!]) & \text{Rule 4.2} \\ [\![\Gamma \rhd N: A \multimap B]\!] & = & [\![\Gamma \rhd \lambda x: A.Nx: A \multimap B]\!]. & \end{array}$$

This is known as a η-rule in λ-calculus nomenclature. It provides the new equation in context

$$\frac{\Gamma \rhd N: A \multimap B}{\Gamma \rhd \lambda x: A.Nx = N: A \multimap B} \multimap_\eta.$$

Essentially we have shown that there exists a natural isomorphism between the following maps

$$\frac{\Gamma \bullet A \to B}{\overline{\Gamma \to A \multimap B,}}$$

thus $\multimap$ provides us with a closed structure on the category corresponding to the bifunctor $\bullet$. Hence we extend our categorical model to a symmetric monoidal *closed* category, $(\mathbb{C}, \bullet, 1, \multimap, \alpha, \lambda, \rho, \gamma)$.

Tensor

The introduction rule for Tensor is of the form

$$\frac{\Gamma \rhd M\colon A \qquad \Delta \rhd N\colon B}{\Gamma, \Delta \rhd M{\otimes}N\colon A{\otimes}B}\,(\otimes_{\mathcal{I}}).$$

To interpret this rule we need a natural transformation with components

$$\Phi_{\Gamma,\Delta}\colon \mathbb{C}(\Gamma, A) \times \mathbb{C}(\Delta, B) \to \mathbb{C}((\Gamma,\Delta), A{\otimes}B).$$

Given morphisms $e\colon\Gamma \to A$, $f\colon\Delta \to B$, $c\colon\Gamma' \to \Gamma$ and $d\colon\Delta' \to \Delta$, naturality gives the equation

$$\mathsf{Split}(\Gamma', \Delta'); c \bullet d; \mathsf{Join}(\Gamma, \Delta); \Phi_{\Gamma,\Delta}(e, f) \quad = \quad \Phi_{\Gamma',\Delta'}((c; e), (d; f)).$$

In particular if we take e to be id_A, f to be id_B, c to be a morphism $m\colon\Gamma \to A$ and d to be a morphism $n\colon\Delta \to B$ then by naturality we have

$$\mathsf{Split}(\Gamma, \Delta); m \bullet n; \Phi_{A,B}(\mathsf{id}_A, \mathsf{id}_B) \quad = \quad \Phi_{\Gamma,\Delta}(m, n).$$

If we define the morphism $\otimes \stackrel{\text{def}}{=} \Phi_{A,B}(\mathsf{id}_A, \mathsf{id}_B)$ then we can make the definition

$$[\![\Gamma, \Delta \rhd M{\otimes}N\colon A{\otimes}B]\!] \quad \stackrel{\text{def}}{=} \quad \mathsf{Split}(\Gamma, \Delta); [\![\Gamma \rhd M\colon A]\!] \bullet [\![\Delta \rhd N\colon B]\!]; \otimes. \tag{4.6}$$

The elimination rule for the tensor is of the form

$$\frac{\Delta \rhd M\colon A{\otimes}B \qquad \Gamma, x\colon A, y\colon B \rhd N\colon C}{\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N\colon C}\,(\otimes_{\mathcal{E}}).$$

This suggests a natural transformation with components

$$\Psi_{\Delta,\Gamma}\colon \mathbb{C}(\Delta, A{\otimes}B) \times \mathbb{C}(\Gamma \bullet A \bullet B, C) \to \mathbb{C}((\Gamma,\Delta), C).$$

Taking morphisms $e\colon\Delta \to A{\otimes}B$, $f\colon\Gamma \bullet A \bullet B \to C$, $c\colon\Delta' \to \Delta$ and $d\colon\Gamma' \to \Gamma$, applying naturality gives the equation

$$\mathsf{Split}(\Gamma', \Delta'); d \bullet c; \mathsf{Join}(\Gamma, \Delta); \Psi_{\Delta,\Gamma}(e, f) \quad = \quad \Psi_{\Delta',\Gamma'}((c; e), ((d \bullet \mathsf{id}_A \bullet \mathsf{id}_B); f)).$$

In particular if we take e to be $\mathsf{id}_{A\otimes B}$, f to be some morphism $n\colon\Gamma \bullet A \bullet B \to C$, c to be some morphism $m\colon\Delta \to A{\otimes}B$ and d to be id_Γ, then by naturality we have

$$\mathsf{Split}(\Gamma, \Delta); \mathsf{id}_\Gamma \bullet m; \Psi_{A\otimes B,\Gamma}(\mathsf{id}_{A\otimes B}, n) \quad = \quad \Psi_{\Delta,\Gamma}(m, n).$$

Thus $\Psi_{\Delta,\Gamma}(m, n)$ can be expressed as the composition $\mathsf{Split}(\Gamma, \Delta); \mathsf{id}_\Gamma \bullet m; \Upsilon_\Gamma(n)$ where Υ is a natural transformation with components

$$\Upsilon_\Gamma\colon \mathbb{C}(\Gamma \bullet A \bullet B, C) \to \mathbb{C}(\Gamma \bullet A{\otimes}B, C).$$

We shall write $(-)^*$ for the effect of this natural transformation and so we can make the definition

$$[\![\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N\colon C]\!] \quad \stackrel{\text{def}}{=} \quad \mathsf{Split}(\Gamma, \Delta); \mathsf{id}_\Gamma \bullet [\![\Delta \rhd M\colon A{\otimes}B]\!]; ([\![\Gamma, x\colon A, y\colon B \rhd N\colon C]\!])^* \tag{4.7}$$

We shall make the simplifying assumption that $(-)^*\colon \mathbb{C}(\Gamma \bullet A \bullet B, C) \to \mathbb{C}(\Gamma \bullet A{\otimes}B, C)$ is natural in C. Thus for morphisms $m\colon\Gamma \bullet A \bullet B \to C$ and $n\colon\Delta \bullet C \to C'$, we have the equality

$$\mathsf{Split}(\Delta,(\Gamma,A{\otimes}B));\mathsf{id}_\Delta\bullet(m)^*;n \quad=\quad (\mathsf{Split}(\Delta,(\Gamma,A,B));\mathsf{id}_\Delta\bullet m;n)^*. \tag{4.8}$$

At the level of terms, this produces the equation in context

$$\frac{\Delta \triangleright M\colon A{\otimes}B \qquad \Gamma,x\colon A,y\colon B\triangleright N\colon C \qquad \Theta,z\colon C\triangleright P\colon D}{\Theta,\Gamma,\Delta\triangleright P[z:=\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ N]=\mathsf{let}\ M\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ P[z:=N]\colon C}\ {\otimes}_{nat}.$$

We have the following equation in context for the tensor

$$\frac{\Gamma\triangleright M\colon A \qquad \Delta\triangleright N\colon B \qquad \Theta,x\colon A,y\colon B\triangleright P\colon C}{\Theta,\Gamma,\Delta\triangleright \mathsf{let}\ M{\otimes}N\ \mathsf{be}\ x{\otimes}y\ \mathsf{in}\ P=P[x:=M,y:=N]\colon C}\ {\otimes}_{Eq}.$$

If we take morphisms $m\colon\Gamma\to A$, $n\colon\Delta\to B$ and $p\colon\Theta\bullet A\bullet B\to C$, this rule amounts to the equation

$$\mathsf{Split}(\Theta,(\Gamma,\Delta));\mathsf{id}_\Theta\bullet\mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Theta\bullet(m\bullet n);\mathsf{id}_\Theta\bullet\otimes;p^* \quad=\quad \mathsf{Split}_3(\Theta,\Gamma,\Delta);\mathsf{id}_\Theta\bullet m\bullet n;p \tag{4.9}$$

or diagrammatically

$$\begin{array}{ccccc}
\Theta\bullet(\Gamma\bullet\Delta) & \xrightarrow{\mathsf{id}_\Theta\bullet(m\bullet n)} & \Theta\bullet(A\bullet B) & \xrightarrow{\mathsf{id}_\Theta\bullet\otimes} & \Theta\bullet A{\otimes}B \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle p^*} \\
(\Theta\bullet\Gamma)\bullet\Delta & \xrightarrow[(\mathsf{id}_\Theta\bullet m)\bullet n]{} & (\Theta\bullet A)\bullet B & \xrightarrow[p]{} & C.
\end{array}$$

We shall make the simplifying assumption that the factorization suggested above is unique, namely that a morphism $p\colon\Theta\bullet A\bullet B\to C$ can be factored *uniquely* as

$$p \quad=\quad \alpha^{-1}_{\Theta,A,B};\mathsf{id}_\Theta\bullet\otimes;p^*. \tag{4.10}$$

In particular, let us consider the case when $p=\otimes$ and Θ is empty. We have that $\otimes;(\otimes)^*=\otimes$, and from our uniqueness assumption we can conclude that

$$(\otimes)^* \quad=\quad \mathsf{id}_{A\otimes B}. \tag{4.11}$$

Using the morphism $\otimes$ we can define a natural transformation $(-)_*\colon\mathbb{C}(\Gamma\bullet A{\otimes}B,C)\to\mathbb{C}(\Gamma\bullet A\bullet B,C)$ by $g\mapsto\alpha^{-1}_{\Gamma,A,B};\mathsf{id}_\Gamma\bullet\otimes;g$. Let us consider whether this is an inverse to the $(-)^*$ transformation.

If we take a morphism $g\colon\Gamma\bullet A{\otimes}B\to C$ then

$$\begin{array}{rcll}
((g)_*)^* & = & (\alpha^{-1}_{\Gamma,A,B};\mathsf{id}_\Gamma\bullet\otimes;g)^* & \\
& = & \mathsf{id}_\Gamma\bullet(\otimes)^*;g & \text{Rule 4.8} \\
& = & g & \text{Rule 4.11.}
\end{array}$$

If we take a morphism $f\colon\Gamma\bullet A\bullet B\to C$ then

$$\begin{array}{rcll}
((f)^*)_* & = & \alpha^{-1}_{\Gamma,A,B};\mathsf{id}_\Gamma\bullet\otimes;f^* & \\
& = & f & \text{Rule 4.10.}
\end{array}$$

Thus we have a natural isomorphism between the maps

$$\frac{\Gamma\bullet A\bullet B\to C}{\Gamma\bullet A{\otimes}B\to C.}$$

Hence we shall assume that $\bullet$ coincides with $\otimes$. We know from above that given a morphism $n: \Gamma \bullet A \otimes B \to C$, the equality $(\alpha^{-1}_{\Gamma,A,B}; \mathsf{id}_\Gamma \bullet \otimes; n)^* = n$ holds. If we precompose with the composite $\mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma \bullet m$ given a morphism $m: \Delta \to A \otimes B$, then we have

$$\begin{array}{rcl}
\mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma \bullet m; (\alpha^{-1}_{\Gamma,A,B}; \mathsf{id}_\Gamma \bullet \otimes; n)^* & = & \mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma \bullet m; n \\
\mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma \bullet m; (\alpha^{-1}_{\Gamma,A,B}; \mathsf{id}_\Gamma \bullet [\![x: A, y: B \rhd x \otimes y: A \otimes B]\!]; [\![\Gamma, z: A \otimes B \rhd N: C]\!])^* & = & \mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma \bullet m; n \\
\mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma \bullet m; ([\![\Gamma, x: A, y: B \rhd N[z := x \otimes y]: C]\!])^* & = & \mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma \bullet m; n \\
\mathsf{Split}(\Gamma,\Delta); \mathsf{id} \bullet [\![\Delta \rhd M: A \otimes B]\!]; ([\![\Gamma, x: A, y: B \rhd N[z := x \otimes y]: C]\!])^* & = & [\![\Gamma, \Delta \rhd N[z := M]: C]\!] \\
[\![\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ N[z := x \otimes y]: C]\!] & = & [\![\Gamma, \Delta \rhd N[z := M]: C]\!].
\end{array}$$

This represents another η-rule and gives the equation in context

$$\frac{\Delta \rhd M: A \otimes B \qquad \Gamma, z: A \otimes B \rhd N: C}{\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ N[z := x \otimes y] = N[z := M]: C} \otimes_\eta.$$

Unit

The introduction rule for the unit is of the form

$$\frac{}{\rhd *: I} (I_\mathcal{I}).$$

We interpret this rule with a unique map $\langle\rangle: 1 \to I$. The elimination rule for the unit is of the form

$$\frac{\Gamma \rhd N: A \qquad \Delta \rhd M: I}{\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N: A} (I_\mathcal{E}).$$

To interpret this rule we need a natural transformation with components

$$\Phi_{\Gamma,\Delta}: \mathbb{C}(\Gamma, A) \times \mathbb{C}(\Delta, I) \to \mathbb{C}((\Gamma,\Delta), A).$$

Given morphisms $e: \Gamma \to A$, $f: \Delta \to I$, $c: \Gamma' \to \Gamma$ and $d: \Delta' \to \Delta$, naturality gives the equation

$$\mathsf{Split}(\Gamma', \Delta'); c \bullet d; \mathsf{Join}(\Gamma,\Delta); \Phi_{\Gamma,\Delta}(e, f) \quad = \quad \Phi_{\Gamma',\Delta'}((c; e), (d; f)).$$

In particular if we take e to be id_A, f to be id_I, c to be a morphism $m: \Gamma \to A$ and d to be a morphism $n: \Delta \to I$, then by naturality we have

$$\mathsf{Split}(\Gamma,\Delta); m \bullet n; \Phi_{A,I}(\mathsf{id}_A, \mathsf{id}_I) \quad = \quad \Phi_{\Gamma,\Delta}(m, n).$$

If we define $\phi \stackrel{\text{def}}{=} \Phi_{A,I}(\mathsf{id}_A, \mathsf{id}_I)$, then we can make the definition

$$[\![\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N: A]\!] \quad \stackrel{\text{def}}{=} \quad \mathsf{Split}(\Gamma,\Delta); [\![\Gamma \rhd N: A]\!] \bullet [\![\Delta \rhd M: I]\!]; \phi. \tag{4.12}$$

We shall make the simplifying assumption that $\phi: A \bullet I \to A$ is natural in A. Taking a morphism $f: \Gamma \bullet A \to B$ this gives

$$\alpha^{-1}_{\Gamma,A,I}; \mathsf{id}_\Gamma \bullet \phi_A; f \quad = \quad f \bullet \mathsf{id}_I; \phi_B.$$

At the level of terms this gives the equation in context

$$\frac{\Gamma \rhd N: A \qquad \Delta \rhd M: I \qquad \Theta, z: A \rhd P: B}{\Theta, \Gamma, \Delta \rhd P[z := \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N] = \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ P[z := N]: B} I_{nat}.$$

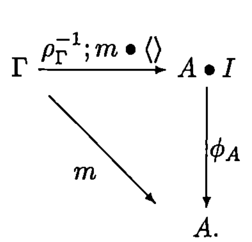
We have the following equation in context for the unit

$$\frac{\rhd *: I \qquad \Gamma \rhd M: A}{\Gamma \rhd \mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ M = M: A}\ I_{Eq}.$$

If we take morphisms $\langle\rangle: 1 \to I$ and $m: \Gamma \to A$, this rule amounts to the condition

$$\rho_\Gamma^{-1}; m \bullet \langle\rangle; \phi_A \quad = \quad m. \tag{4.13}$$

As with the case for the Tensor, we shall assume that this factorization is *unique*. Diagrammatically



In particular, let us take the case when $m = \phi_A$. We have that $\rho^{-1}; \phi \bullet \langle\rangle; \phi = \phi$, and from our uniqueness assumption we can conclude

$$\rho_{A \bullet I}^{-1}; \phi_A \bullet \langle\rangle \quad = \quad \mathsf{id}_{A \bullet I}. \tag{4.14}$$

We can formulate two natural transformations,

$$\begin{array}{l} (-)^*: \mathbb{C}(\Gamma \bullet I, A) \to \mathbb{C}(\Gamma \bullet 1, A) \\ f \mapsto \mathsf{id}_\Gamma \bullet \langle\rangle; f, \end{array}$$

and

$$\begin{array}{l} (-)_*: \mathbb{C}(\Gamma \bullet 1, A) \to \mathbb{C}(\Gamma \bullet I, A) \\ g \mapsto \phi_\Gamma; \rho_\Gamma^{-1}; g. \end{array}$$

Let us consider whether these are inverses to each other. First take a morphism $f: \Gamma \bullet I \to A$,then

$$\begin{array}{rcll} ((f)^*)_* & = & (\mathsf{id}_\Gamma \bullet \langle\rangle; f)_* & \\ & = & \phi_\Gamma; \rho^{-1}; \mathsf{id}_\Gamma \bullet \langle\rangle; f & \\ & = & \rho_{\Gamma \bullet 1}^{-1}; \phi_\Gamma \bullet \mathsf{id}_1; \mathsf{id}_\Gamma \bullet \langle\rangle; f & \\ & = & \rho_{\Gamma \bullet 1}^{-1}; \phi_\Gamma \bullet \langle\rangle; f & \\ & = & f & \text{Rule 4.14.} \end{array}$$

If we take a morphism $g: \Gamma \bullet 1 \to A$ then

$$\begin{array}{rcll} ((g)_*)^* & = & (\phi; \rho^{-1}; g)^* & \\ & = & \mathsf{id}_\Gamma \bullet \langle\rangle; \phi; \rho^{-1}; g & \\ & = & \rho; \mathsf{id}; \rho^{-1}; g & \text{Rule 4.13} \\ & = & g. & \end{array}$$

Thus we have a natural isomorphism between the maps

$$\frac{\Gamma \bullet I \to A}{\Gamma \bullet 1 \to A.}$$

Hence, as for Tensor, we can assume that I coincides with 1. We know from above that given a morphism $n: \Gamma \bullet I \to A$, the equality $\phi; \rho^{-1}; \mathsf{id}_\Gamma \bullet \langle\rangle; n = n$ holds. Precomposing with $\mathsf{Split}(\Gamma, \Delta); \mathsf{id}_\Gamma \bullet m$, given a morphism $m: \Delta \to I$ gives

$$\begin{array}{rcl} \mathsf{Split}(\Gamma, \Delta); \mathsf{id}_\Gamma \bullet m; \phi; \rho^{-1}; \mathsf{id}_\Gamma \bullet \langle\rangle; n & = & \mathsf{Split}(\Gamma, \Delta); \mathsf{id}_\Gamma \bullet m; n \\ [\![\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N[z := *]: A]\!] & = & [\![\Gamma, \Delta \rhd N[z := M]: A]\!]. \end{array}$$

This represents another so-called η-rule and gives the equation in context

$$\frac{\Delta \triangleright M{:}\, I \qquad \Gamma, z{:}\, I \triangleright N{:}\, A}{\Gamma, \Delta \triangleright \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N[z := *] = N[z := M]{:}\, A}\ I_\eta.$$

In summary we have seen that $\otimes$ coincides with $\bullet$ and I with 1 and, hence, from now on we shall use the logical symbols for both the logic and the categorical model.

With

In contrast with the problems this connective caused in the proof theoretical study in Chapter 2, it is unproblematic as far as finding a categorical model. The introduction rule for With is

$$\frac{\Gamma \triangleright M{:}\, A \qquad \Gamma \triangleright N{:}\, B}{\Gamma \triangleright \langle M, N\rangle{:}\, A\&B}\ (\&_{\mathcal{I}}).$$

To interpret this rule we need a natural transformation with components

$$\Phi_\Gamma{:}\, \mathbb{C}(\Gamma, A) \times \mathbb{C}(\Gamma, B) \to \mathbb{C}(\Gamma, A\&B).$$

Given morphisms $e{:}\, \Gamma \to A$, $f{:}\, \Gamma \to B$ and $c{:}\, \Gamma' \to \Gamma$, naturality gives the equation

$$c; \Phi_\Gamma(e, f) \quad = \quad \Phi_{\Gamma'}((c; e), (c; f)).$$

In particular if we take c to be $\langle m, n\rangle{:}\, \Gamma \to A \times B$ (where $m{:}\, \Gamma \to A$ and $n{:}\, \Gamma \to B$), e to be $\mathsf{fst}{:}\, A \times B \to A$ and f to be $\mathsf{snd}{:}\, A \times B \to B$ then by naturality we have

$$\langle m, n\rangle; \Phi(\mathsf{fst}, \mathsf{snd}) \quad = \quad \Phi(m, n).$$

If we define the morphism $\mathsf{pair} \stackrel{\text{def}}{=} \Phi(\mathsf{fst}, \mathsf{snd})$ then we can make the definition

$$[\![\Gamma \triangleright \langle M, N\rangle{:}\, A\&B]\!] \quad \stackrel{\text{def}}{=} \quad \langle [\![\Gamma \triangleright M{:}\, A]\!], [\![\Gamma \triangleright M{:}\, B]\!]\rangle; \mathsf{pair}. \tag{4.15}$$

The first elimination rule for the With is of the form

$$\frac{\Gamma \triangleright M{:}\, A\&B}{\Gamma \triangleright \mathsf{fst}(M){:}\, A}\ (\&_{\mathcal{E}-1}).$$

This suggests a natural transformation

$$\Psi{:}\, \mathbb{C}(-, A\&B) \to \mathbb{C}(-, A).$$

However, by the Yoneda lemma [53, Page 61] we know that there is the bijection

$$[\mathbb{C}^{op}, \mathbb{S}\mathrm{ets}](\mathbb{C}(-, A\&B), \mathbb{C}(-, A)) \quad \cong \quad \mathbb{C}(A\&B, A).$$

By actually constructing this isomorphism we find that the components of Ψ are induced by post-composition with a morphism $\pi_1{:}\, A\&B \to A$. Thus we can make the definition

$$[\![\Gamma \triangleright \mathsf{fst}(M){:}\, A]\!] \quad \stackrel{\text{def}}{=} \quad [\![\Gamma \triangleright M{:}\, A\&B]\!]; \pi_1. \tag{4.16}$$

The second elimination rule for With proceeds in a similar fashion and results in the definition (where $\pi_2{:}\, A\&B \to B$).

$$[\![\Gamma \rhd \mathsf{snd}(M) \colon B]\!] \stackrel{\text{def}}{=} [\![\Gamma \rhd M \colon A\&B]\!]; \pi_2. \tag{4.17}$$

We have the following equations in context for With.

$$\frac{\Gamma \rhd M \colon A \qquad \Gamma \rhd N \colon B}{\Gamma \rhd \mathsf{fst}(\langle M, N\rangle) = M \colon A}\,\&_{Eq-1} \qquad \frac{\Gamma \rhd M \colon A \qquad \Gamma \rhd N \colon B}{\Gamma \rhd \mathsf{snd}(\langle M, N\rangle) = N \colon B}\,\&_{Eq-2}$$

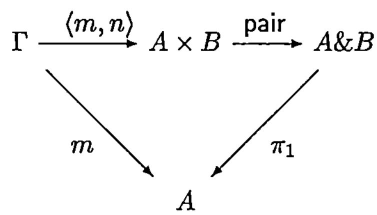
If we take morphisms $m \colon \Gamma \to A$ and $n \colon \Gamma \to B$, these rules amount to the equations

$$\langle m, n\rangle; \mathsf{pair}; \pi_1 \;=\; m, \tag{4.18}$$

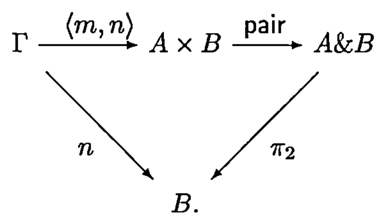
and

$$\langle m, n\rangle; \mathsf{pair}; \pi_2 \;=\; n, \tag{4.19}$$

or diagrammatically



and



We shall make the simplifying assumption that the factorizations suggested above are *unique*. By taking the case where $m = \pi_1$ and $n = \pi_2$, this amounts to the equation

$$\mathsf{id}_{A\&B} \;=\; \langle \pi_1, \pi_2\rangle; \mathsf{pair}. \tag{4.20}$$

We can hence form two operations.

$$\begin{array}{l}(-)^* \colon \mathbb{C}(\Gamma, A\&B) \to \mathbb{C}(\Gamma, A \times B)\\ f \mapsto \langle f; \pi_1, f; \pi_2\rangle,\end{array}$$

and

$$\begin{array}{l}(-)_* \colon \mathbb{C}(\Gamma, A \times B) \to \mathbb{C}(\Gamma, A\&B)\\ g \mapsto g; \mathsf{pair}.\end{array}$$

If we take a morphism $f \colon \Gamma \to A\&B$ then

$$\begin{array}{rcll}((f)^*)_* & = & \langle f; \pi_1, f; \pi_2\rangle; \mathsf{pair} & \\ & = & f; \langle \pi_1, \pi_2\rangle; \mathsf{pair} & \text{Naturality}\\ & = & f & \text{Rule 4.20.}\end{array}$$

If we take morphisms $g \colon \Gamma \to A$ and $h \colon \Gamma \to B$ then

$$\begin{array}{rcll}((\langle m, n\rangle)_*)^* & = & \langle(\langle m, n\rangle; \mathsf{pair}; \pi_1), (\langle m, n\rangle; \mathsf{pair}; \pi_2)\rangle & \\ & = & \langle m, n\rangle & \text{Rules 4.18 and 4.19 .}\end{array}$$

Hence we shall assume that $\times$ coincides with & and, again, we shall use the logical symbol for both the logic and the categorical model. We can precompose the uniqueness equality (Rule 4.20) with a morphism $m\colon\Gamma \to A\&B$ to get

$$\begin{array}{rcl} m; \langle \pi_1, \pi_2\rangle; \mathsf{pair} & = & m \\ m; \langle [\![x\colon A\&B \triangleright \mathsf{fst}(x)\colon A]\!], [\![x\colon A\&B \triangleright \mathsf{snd}(x)\colon B]\!]\rangle; \mathsf{pair} & = & m \\ [\![\Gamma \triangleright M\colon A\&B]\!]; [\![x\colon A\&B \triangleright \langle \mathsf{fst}(x), \mathsf{snd}(x)\rangle\colon A\&B]\!] & = & [\![\Gamma \triangleright M\colon A\&B]\!] \\ [\![\Gamma \triangleright \langle \mathsf{fst}(M), \mathsf{snd}(M)\rangle\colon A\&B]\!] & = & [\![\Gamma \triangleright M\colon A\&B]\!]. \end{array}$$

This represents another η-rule and gives the equation in context

$$\frac{\Gamma \triangleright M\colon A\&B}{\Gamma \triangleright \langle \mathsf{fst}(M), \mathsf{snd}(M)\rangle = M\colon A\&B}\ \&_\eta.$$

The Additive Disjunction

The (first) introduction rule for the additive disjunction is of the form

$$\frac{\Gamma \triangleright M\colon A}{\Gamma \triangleright \mathsf{inl}(M)\colon A \oplus B}\ \oplus_{\mathcal{I}-1}.$$

To interpret this rule we need a natural transformation

$$\Phi\colon \mathbb{C}(-, A) \to \mathbb{C}(-, A \oplus B).$$

Again by the Yoneda lemma [53, Page 61] we know that there is the bijection

$$[\mathbb{C}^{op}, \mathbb{S}\mathrm{ets}](\mathbb{C}(-,A), \mathbb{C}(-, A \oplus B) \cong \mathbb{C}(A, A \oplus B).$$

By actually constructing this isomorphism we find that the components of Φ are induced by post-composition with a morphism $i_{A\oplus B}\colon A \to A \oplus B$. Thus we can make the definition

$$[\![\Gamma \triangleright \mathsf{inl}(M)\colon A \oplus B]\!] \stackrel{\text{def}}{=} [\![\Gamma \triangleright M\colon A]\!]; i_{A\oplus B}. \tag{4.21}$$

The second introduction rule for the additive disjunction can be modelled in a similar way and we arrive at the definition (where $j_{A\oplus B}\colon B \to A \oplus B$)

$$[\![\Gamma \triangleright \mathsf{inr}(M)\colon A \oplus B]\!] \stackrel{\text{def}}{=} [\![\Gamma \triangleright M\colon B]\!]; j_{A\oplus B}. \tag{4.22}$$

The elimination rule is of the form

$$\frac{\Delta \triangleright M\colon A \oplus B \qquad \Gamma, x\colon A \triangleright N\colon C \qquad \Gamma, y\colon B \triangleright P\colon C}{\Gamma, \Delta \triangleright \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P\colon C}\ (\oplus_\mathcal{E}).$$

To interpret this rule we need a natural transformation with components

$$\Psi_{\Gamma,\Delta}\colon \mathbb{C}(\Delta, A \oplus B) \times \mathbb{C}(\Gamma \otimes A, C) \times \mathbb{C}(\Gamma \otimes B, C) \to \mathbb{C}((\Gamma, \Delta), C).$$

Given morphisms $e\colon \Delta \to A \oplus B$, $f\colon \Gamma\otimes A \to C$, $g\colon \Gamma\otimes B \to C$, $c\colon \Delta' \to \Delta$ and $d\colon \Gamma' \to \Gamma$, naturality gives the equation

$$\mathsf{Split}(\Gamma', \Delta'); d\otimes c; \mathsf{Join}(\Gamma, \Delta); \Psi_{\Gamma,\Delta}(e, f, g) \;=\; \Psi_{\Gamma',\Delta'}((c; e), (d; f), (d; g)).$$

In particular if we take c to be a morphism $m\colon \Delta \to A\otimes B$, e to be $\mathsf{id}_{A\oplus B}$, f to be some morphism $n\colon \Gamma\otimes A \to C$, g to be some morphism $p\colon \Gamma\otimes B \to C$ and d to be id_Γ, then by naturality we have

$$\mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Gamma\otimes m;\Psi(\mathsf{id}_{A\oplus B},n,p) \quad = \quad \Psi(m,n,p).$$

Thus $\Psi(m,n,p)$ can be expressed as the composite $\mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Gamma\otimes m;\Omega(n,p)$ where Ω is the natural transformation with components

$$\Omega_\Gamma\colon \mathbb{C}(\Gamma\otimes A, C)\times\mathbb{C}(\Gamma\otimes B, C)\to\mathbb{C}(\Gamma\otimes(A\oplus B),C).$$

We shall make the simplifying assumption that Ω is natural in C. Taking morphisms $e\colon\Gamma\otimes A\to C$, $f\colon\Gamma\otimes B\to C$ and $g\colon\Delta\otimes C\to D$, this amounts to

$$\mathsf{Split}(\Delta,(\Gamma,A\otimes B));\mathsf{id}_\Delta\otimes\Omega(e,f);g \quad = \quad \Omega((\mathsf{Split}(\Delta,(\Gamma,A));\mathsf{id}_\Delta\otimes e;g),(\mathsf{Split}(\Delta,(\Gamma,B));\mathsf{id}_\Delta\otimes f;g)).$$

In particular if we take e to be $\mathsf{inl}\colon\Gamma\otimes A\to(\Gamma\otimes A)\oplus(\Gamma\otimes B)$, f to be $\mathsf{inr}\colon\Gamma\otimes B\to(\Gamma\otimes A)\oplus(\Gamma\otimes B)$, and g to be $[m,n]$ where $m\colon\Gamma\otimes A\to C$ and $n\colon\Gamma\otimes B\to C$, then naturality amounts to

$$\Omega_\Gamma(\mathsf{inl},\mathsf{inr});[m,n] \quad = \quad \Omega(e,f).$$

Thus writing dist_Γ for the morphism $\Omega_\Gamma(\mathsf{inl},\mathsf{inr})$, we can make the definition

$$\begin{array}{l}[\![\Gamma,\Delta\rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N \parallel \mathsf{inr}(y)\ \to\ P\colon C]\!] \stackrel{\mathrm{def}}{=} \\ \qquad \mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Gamma\otimes[\![\Delta\rhd M\colon A\oplus B]\!];\mathsf{dist}_\Gamma;[[\![\Gamma,x\colon A\rhd N\colon C]\!],[\![\Gamma,y\colon B\rhd P\colon C]\!]].\end{array} \tag{4.23}$$

The naturality of the coproduct construction gives rise to the equation in context

$$\frac{\Delta\rhd M\colon A\oplus B \quad \Gamma,x\colon A\rhd N\colon C \quad \Gamma,y\colon B\rhd P\colon C \quad \Theta,z\colon C\rhd Q\colon D}{\begin{array}{lll}\Theta,\Gamma,\Delta & \rhd & Q[z:=\mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N\parallel\mathsf{inr}(y)\ \to\ P] \\ & & = \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ (Q[z:=N])\parallel\mathsf{inr}(y)\ \to\ (Q[z:=P])\colon D\end{array}}\oplus_{nat}.$$

We have the following two equations in context for the additive disjunction

$$\frac{\Delta\rhd M\colon A \qquad \Gamma,x\colon A\rhd N\colon C \qquad \Gamma,y\colon B\rhd P\colon C}{\Gamma,\Delta\rhd\mathsf{case}\ \mathsf{inl}(M)\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N\parallel\mathsf{inr}(y)\ \to\ P=N[x:=M]\colon C}\oplus_{Eq-1},$$

and

$$\frac{\Delta\rhd M\colon A \qquad \Gamma,x\colon A\rhd N\colon C \qquad \Gamma,y\colon B\rhd P\colon C}{\Gamma,\Delta\rhd\mathsf{case}\ \mathsf{inr}(M)\ \mathsf{of}\ \mathsf{inl}(x)\ \to\ N\parallel\mathsf{inr}(y)\ \to\ P=P[y:=M]\colon C}\oplus_{Eq-2}.$$

Taking morphisms $m\colon\Delta\to A\oplus B$, $n\colon\Gamma\otimes A\to C$ and $p\colon\Gamma\otimes B\to C$ these amount to the equations

$$\mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Gamma\otimes(m;i);\mathsf{dist};[n,p] \quad = \quad \mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Gamma\otimes m;n,$$

and

$$\mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Gamma\otimes(m;j);\mathsf{dist};[n,p] \quad = \quad \mathsf{Split}(\Gamma,\Delta);\mathsf{id}_\Gamma\otimes m;p.$$

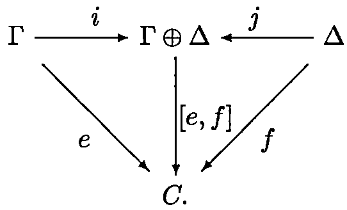
We shall make the simplifying assumption that the factorizations suggested above are unique and thus given morphisms $e\colon\Gamma\to C$ and $f\colon\Delta\to C$, the following equalities hold

$$i;[e,f] \quad = \quad e,$$

and

$$j;[e,f] \quad = \quad f,$$

or diagrammatically



Thus we shall model the additive disjunction with a *coproduct* and hence the morphisms i and j are modelled by the injection morphisms inl and inr respectively.

Taking the uniqueness assumption from above, we derive the equality

$$\mathsf{id}_{A\oplus B} \quad = \quad [\mathsf{inl}, \mathsf{inr}].$$

Precomposing this with a morphism $m: \Gamma \to A \oplus B$ we get

$$\begin{array}{rcl} m; [\mathsf{inl}, \mathsf{inr}] & = & m \\ m; [[\![x: A \rhd \mathsf{inl}(x): A \oplus B]\!], [\![y: B \rhd \mathsf{inr}(y): A \oplus B]\!]] & = & m \\ [\![\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \ \to\ \mathsf{inl}(x) \parallel \mathsf{inr}(y) \ \to\ \mathsf{inr}(y): A \oplus B]\!] & = & [\![\Gamma \rhd M: A \oplus B]\!]. \end{array}$$

This represents another η-rule and gives the equation in context

$$\frac{\Gamma \rhd M: A \oplus B}{\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \ \to\ \mathsf{inl}(x) \parallel \mathsf{inr}(y) \ \to\ \mathsf{inr}(y) = M: A \oplus B}\ \oplus_\eta.$$

The Additive Units

The introduction rule for $\mathbf{t}$ is of the form

$$\frac{\Gamma_1 \rhd M_1: A_1 \quad \cdots \quad \Gamma_n \rhd M_n: A_n}{\Gamma_1, \ldots, \Gamma_n \rhd \mathsf{true}(\vec{M}): \mathbf{t}}\ (\mathbf{t}_\mathcal{I}).$$

To interpret this rule we need a natural transformation with components

$$\Phi_{\Gamma_1,\ldots,\Gamma_n}: \mathbb{C}(\Gamma_1, A_1) \times \cdots \times \mathbb{C}(\Gamma_n, A_n) \to \mathbb{C}((\Gamma_1, \ldots, \Gamma_n), \mathbf{t}).$$

Given morphisms $e_i: \Gamma_i \to A_i$ and $c_i: \Gamma'_i \to \Gamma_i$, then naturality gives the equality

$$\begin{array}{l} \mathsf{Split_n}(\Gamma'_1, \ldots, \Gamma'_n); c_1 \otimes \cdots \otimes c_n; \mathsf{Join_n}(\Gamma_1, \ldots, \Gamma_n); \Phi(e_1, \ldots, e_n) \\ = \Phi((c_1; e_1), \ldots, (c_n; e_n)). \end{array}$$

If we take c_i to be morphisms $m_i: \Gamma_i \to A_i$ and e_i to be id_{A_i}, then by naturality we have

$$\mathsf{Split_n}(\Gamma_1, \ldots, \Gamma_n); m_1 \otimes \cdots \otimes m_n; \Phi(\mathsf{id}_{A_1}, \ldots, \mathsf{id}_{A_n}) \quad = \quad \Phi(m_1, \ldots, m_n).$$

We shall write $\top$ in place of $\Phi(\mathsf{id}_{A_1}, \ldots, \mathsf{id}_{A_n}): A_1 \otimes \cdots \otimes A_n \to \mathbf{t}$. We have little else to guide us and we shall make the simplifying assumption that $\mathbf{t}$ is a *terminal object* and thus $\top$ is the terminal morphism.

$$[\![\Gamma \rhd \mathsf{true}(\vec{M}): \mathbf{t}]\!] \quad \stackrel{\text{def}}{=} \quad \top_\Gamma. \tag{4.24}$$

The elimination rule for $\mathbf{f}$ is as follows

$$\frac{\Gamma_1 \rhd M_1 : A_1 \quad \cdots \quad \Gamma_n \rhd M_n : A_n \qquad \Delta \rhd N : \mathbf{f}}{\Gamma_1, \ldots, \Gamma_n, \Delta \rhd \mathsf{false}_B(\vec{M}\ ;\ N) : B}\ (\mathbf{f}_{\mathcal{E}}).$$

To interpret this rule we need a natural transformation with components

$$\Phi_{\Gamma_1,\ldots,\Gamma_n,\Delta} : \mathbb{C}(\Gamma_1, A_1) \times \cdots \times \mathbb{C}(\Gamma_n, A_n) \times \mathbb{C}(\Delta, \mathbf{f}) \to \mathbb{C}((\Gamma_1, \cdots, \Gamma_n, \Delta), B).$$

Given morphisms $e_i : \Gamma_i \to A_i$, $c_i : \Gamma_i' \to \Gamma_i$, $g : \Delta \to \mathbf{f}$ and $d : \Delta' \to \Delta$, naturality gives

$$\mathsf{Split}_{n+1}(\Gamma_1', \ldots, \Gamma_n', \Delta'); c_1 \otimes \cdots \otimes c_n \otimes d; \mathsf{Join}_{n+1}(\Gamma_1, \ldots, \Gamma_n, \Delta); \Phi_{\Gamma_1,\ldots,\Gamma_n,\Delta}(e_1, \ldots, e_n, g) = \Phi_{\Gamma_1',\ldots,\Gamma_n',\Delta'}((c_1; e_1), \ldots, (c_n; e_n), (d; g)).$$

In particular if we take c_i to be morphisms $m_i : \Gamma_i \to A_i$, d to be a morphism $p : \Delta \to \mathbf{f}$, e_i to be id_{A_i} and g to be $\mathsf{id}_{\mathbf{f}}$, then by naturality we have

$$\mathsf{Split}_{n+1}(\Gamma_1, \ldots, \Gamma_n; \Delta); m_1 \otimes \cdots \otimes m_n \otimes p; \Phi(\mathsf{id}_{A_1}, \ldots, \mathsf{id}_{A_n}, \mathsf{id}_{\mathbf{f}}) \;=\; \Phi(m_1, \ldots, m_n, p). \quad (4.25)$$

Note that there can be zero m terms, in which case we would have

$$p; \Phi_{\mathbf{f}}(\mathsf{id}_{\mathbf{f}}) \;=\; \Phi_{\Delta}(p). \quad (4.26)$$

We shall write $\bot_B$ for the morphism $\Phi_{\mathbf{f}}(\mathsf{id}_{\mathbf{f}}) : \mathbf{f} \to B$. We recall from our analysis of the additive disjunction, that we introduced a natural transformation $\mathsf{dist}_\Gamma : \Gamma \otimes (A \oplus B) \to (\Gamma \otimes A) \oplus (\Gamma \otimes B)$.[3] As we are dealing with the nullary version of this connective, we shall introduce a natural transformation $\mathsf{ndist}_\Gamma : \Gamma \otimes \mathbf{f} \to \mathbf{f}$. Hence we shall make the definition

$$\begin{aligned}[\![\Gamma_1, \ldots, \Gamma_n, \Delta \rhd \mathsf{false}_B(\vec{M}\ ;\ N) : B]\!] \stackrel{\text{def}}{=}\ & \mathsf{Split}_{n+1}(\Gamma_1, \ldots, \Gamma_n; \Delta); \\ & [\![\Gamma_1 \rhd M_1 : A_1]\!] \otimes \cdots \otimes [\![\Gamma_n \rhd M_n : A_n]\!] \otimes [\![\Delta \rhd N : \mathbf{f}]\!]; \\ & \mathsf{ndist}_{\Gamma_1 \otimes \ldots \otimes \Gamma_n}; \bot_B. \end{aligned} \quad (4.27)$$

An important question is whether we take $\mathbf{f}$ to be the *initial object* (and hence $\bot$ to be the initial morphism). Certainly there is nothing so far to suggest this categorically appealing assumption. Indeed, the similar assumption for categorical models of **IL** is somewhat controversial, and, as shown by Harnik and Makkai [37], ensures that certain well-known properties fail. In particular they quote the abstract result of Lambek and Scott [52, Page 67] that in a bicartesian closed category[4] there is at most one morphism $A \to \mathbf{f}$ for any object A. Thus if we define negation of a proposition A as $A \supset \mathbf{f}$, this means that no matter how many proofs we have of a proposition A, all the proofs of its negation $\neg A$ (and also its double negation) will be collapsed as a single morphism in the categorical model. However, taking the reductions from Chapter 3 and in particular the commuting conversions, we find that we need to model equations in context such as

$$\frac{\Delta \rhd N : \mathbf{f}}{\Delta \rhd \mathsf{fst}(\mathsf{false}_{A\&B}(\_;\ N)) = \mathsf{false}_A(\_;\ N) : A}.$$

Categorically this amounts to the equality

$$n; \bot_{A\&B}; \mathsf{fst} \;=\; n; \bot_A.$$

Certainly if we require our categorical model to model *all* the reductions, including the commuting conversions, then we need the assumption of initiality.[5] With this motivation we shall make the simplifying assumption that $\mathbf{f}$ is the initial object.

[3] In fact, abstract reasoning tells us that this is a natural *isomorphism*.

[4] A *bicartesian closed category* is a CCC with coproducts and an initial object.

[5] This is also the case for **IL**.

The Exponential

The rule for *Promotion* is of the form

$$\frac{\Gamma_1 \triangleright M_1 : !A_1 \quad \cdots \quad \Gamma_n \triangleright M_n : !A_n \qquad x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B}{\Gamma_1, \ldots, \Gamma_n \triangleright \mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N : !B}\ \textit{Promotion.}$$

To interpret this rule we need a natural transformation with components

$$\Phi_{\Gamma_1,\ldots,\Gamma_n} : \mathbb{C}(\Gamma_1, !A_1) \times \ldots \times \mathbb{C}(\Gamma_n, !A_n) \times \mathbb{C}(!A_1 \otimes \ldots \otimes !A_n, B) \to \mathbb{C}((\Gamma_1, \ldots, \Gamma_n), !B).$$

Given morphisms $e_i : \Gamma_i \to !A_i$, $c_i : \Gamma_i' \to \Gamma_i$ and $d : !A_1 \otimes \ldots \otimes !A_n \to B$, naturality gives the equation

$$\begin{aligned}&\mathsf{Splitn}(\Gamma_1', \ldots, \Gamma_n'); c_1 \otimes \ldots \otimes c_n; \mathsf{Joinn}(\Gamma_1, \ldots, \Gamma_n); \Phi_{\Gamma_1,\ldots,\Gamma_n}(e_1, \ldots, e_n, d) \\ &= \Phi_{\Gamma_1',\ldots,\Gamma_n'}((c_1; e_1), \ldots, (c_n; e_n), d).\end{aligned}$$

In particular if we take c_i to be morphisms $m_i : \Gamma_i \to !A_i$, e_i to be $\mathsf{id}_{!A_i}$ and d to be morphism $p : !A_1 \otimes \ldots \otimes !A_n \to B$, then by naturality we have

$$\mathsf{Splitn}(\Gamma_1, \ldots, \Gamma_n); m_1 \otimes \ldots \otimes m_n; \Phi_{!A_1,\ldots,!A_n}(\mathsf{id}_{!A_1}, \ldots, \mathsf{id}_{!A_n}, p) \quad = \quad \Phi_{\Gamma_1,\ldots,\Gamma_n}(m_1, \ldots, m_n, p).$$

Thus $\Phi(m_1, \ldots, m_n, p)$ can be expressed as the composition $\mathsf{Splitn}(\Gamma_1, \ldots, \Gamma_n); m_1 \otimes \ldots \otimes m_n; \Psi(p)$, where Ψ is a transformation

$$\Psi : \mathbb{C}(!A_1 \otimes \ldots \otimes !A_n, B) \to \mathbb{C}(!A_1 \otimes \ldots \otimes !A_n, !B).$$

We shall write $(-)^*$ for the effect of this transformation and so we can make the preliminary definition

$$\begin{aligned}&[\![\Gamma_1, \ldots, \Gamma_n \triangleright \mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N : !B]\!] \stackrel{\mathrm{def}}{=} \\ &\quad \mathsf{Splitn}(\Gamma_1, \ldots, \Gamma_n); [\![\Gamma_1 \triangleright M_1 : !A_1]\!] \otimes \ldots \otimes [\![\Gamma_n \triangleright M_n : !A_n]\!]; ([\![x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B]\!])^*.\end{aligned}$$

(We shall see later that further analysis will lead us to a more precise definition.) The rule for *Dereliction* is of the form

$$\frac{\Gamma \triangleright M : !A}{\Gamma \triangleright \mathsf{derelict}(M) : A}\ \textit{Dereliction.}$$

To interpret this rule we need a natural transformation

$$\Phi : \mathbb{C}(-, !A) \to \mathbb{C}(-, A).$$

However, by the Yoneda lemma [53, Page 61] we know that there is the bijection

$$[\mathbb{C}^{\mathrm{op}}, \mathbb{S}\mathrm{ets}](\mathbb{C}(-, !A), \mathbb{C}(-, A)) \quad \cong \quad \mathbb{C}(!A, A).$$

By actually constructing this isomorphism, we find that the components of Φ are induced by post-composition with a morphism $\epsilon : !A \to A$. Thus we can make the definition

$$[\![\Gamma \triangleright \mathsf{derelict}(M) : A]\!] \quad \stackrel{\mathrm{def}}{=} \quad [\![\Gamma \triangleright M : !A]\!]; \epsilon. \tag{4.28}$$

We have the following equation in context for *Dereliction*

$$\frac{\Gamma_1 \triangleright M_1 : !A_1 \quad \ldots \quad \Gamma_n \triangleright M_n : !A_n \qquad x_1 : !A_1, \ldots, x_n : !A_n \triangleright N : B}{\begin{aligned}\Gamma_1, \ldots, \Gamma_n \triangleright \quad & \mathsf{derelict}(\mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N) \\ & = N[x_1 := M_1, \ldots, x_n := M_n] : B\end{aligned}}\ \mathit{Der}_{Eq}.$$

If we take morphisms $m_i\colon \Gamma_i \to !A_i$ and $p\colon !A_1\otimes\ldots\otimes !A_n \to B$ this rule amounts to the equality

$$\mathsf{Split}_n(\Gamma_1,\ldots,\Gamma_n); m_1\otimes\ldots\otimes m_n; (p)^*; \epsilon \quad = \quad \mathsf{Split}_n(\Gamma_1,\ldots,\Gamma_n); m_1\otimes\ldots\otimes m_n; p, \qquad (4.29)$$

or diagrammatically,

$$\begin{array}{ccccc} \Gamma_1\otimes\ldots\otimes\Gamma_n & \xrightarrow{m_1\otimes\ldots\otimes m_n} & !A_1\otimes\ldots\otimes !A_n & \xrightarrow{p^*} & !B \\ & & & \searrow_{p} & \downarrow{\scriptstyle\epsilon} \\ & & & & B. \end{array}$$

In accordance with our treatment of Tensor and Unit, we might make the assumption that the factorization suggested above be *unique*. However let us consider an immediate consequence of such an assumption. The equation in context for *Dereliction* gives us that

$$\begin{array}{ccccc} !!A & \xrightarrow{\mathsf{id}_{!!A}} & !!A & \xrightarrow{(!\epsilon_A)^*} & !!A \\ & & & \searrow_{!\epsilon_A} & \downarrow{\scriptstyle\epsilon_{!A}} \\ & & & & !A, \end{array}$$

commutes. An assumption of uniqueness implies that $(!\epsilon_A)^* = \mathsf{id}_{!!A}$. Hence we derive the equality

$$!\epsilon_A \quad = \quad \epsilon_{!A}. \qquad (4.30)$$

Although at the moment this may seem reasonable, we shall see later how such an equality is sufficient to collapse the model so that $!A \cong !!A$. For now, therefore, we shall not demand the uniqueness of the factorization; but rather we shall consider some different constructions.

We can certainly define the operation

$$\begin{array}{l} !\colon \mathbb{C}(\Gamma, B) \to \mathbb{C}(!\Gamma, !B) \\ f \mapsto (\epsilon_\Gamma; f)^*. \end{array}$$

Thus we have an operation of the form

$$\frac{\Gamma \xrightarrow{f} A}{!\Gamma \xrightarrow{!f} !A.}$$

We shall make the simplifying assumption that this operation is a *functor*. However we still need some extra structure. If Γ is represented by the object $A_1\otimes\ldots\otimes A_n$ then $!\Gamma$ will be represented by $!(A_1\otimes\ldots\otimes A_n)$, but clearly by $!\Gamma$ we really mean $!A_1\otimes\ldots\otimes !A_n$.[6] Thus we shall make the additional assumption that ! is a (symmetric) *monoidal* functor. This notion is due to Eilenberg and Kelly [27] (although originally given in the enriched setting) and for completeness we repeat the definition.

Definition 25. (Eilenberg and Kelly) A *symmetric monoidal functor* between SMCs $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$ and $(\mathbb{C}', \bullet', 1', \alpha', \lambda', \rho', \gamma')$ is a functor $F\colon \mathbb{C} \to \mathbb{C}'$ equipped with

[6] This additional complication is essentially due to our use of a tensor product to represent the comma on the left hand side of a sequent. Ideally we should use some form of *multicategory* [51]. Currently this concept is not well developed, although recent work by de Paiva and Ritter [25] looks promising.

1. A morphism $m_{1'}: 1' \to F1$.
2. For any two objects A and B in $\mathbb{C}$, a natural transformation $m_{A,B}: F(A) \bullet' F(B) \to F(A \bullet B)$.

These must satisfy the following diagrams:

$$
\begin{array}{ccc}
F1 \bullet' FA & \xrightarrow{m_{1,A}} & F(1 \bullet A) \\
{\scriptstyle m_{1'} \bullet' \mathsf{id}_{FA}} \uparrow & & \downarrow {\scriptstyle F(\lambda_A)} \\
1' \bullet' FA & \xrightarrow[\lambda'_{FA}]{} & FA
\end{array}
\qquad
\begin{array}{ccc}
FA \bullet' F1 & \xrightarrow{m_{A,1}} & F(A \bullet 1) \\
{\scriptstyle \mathsf{id}_{FA} \bullet' m_{1'}} \uparrow & & \downarrow {\scriptstyle F(\rho_A)} \\
FA \bullet' 1' & \xrightarrow[\rho'_{FA}]{} & FA
\end{array}
$$

$$
\begin{array}{ccccc}
(FA \bullet' FB) \bullet' FC & \xrightarrow{m_{A,B} \bullet' \mathsf{id}_{FC}} & F(A \bullet B) \bullet' FC & \xrightarrow{m_{A \bullet B, C}} & F((A \bullet B) \bullet C) \\
{\scriptstyle \alpha'_{FA,FB,FC}} \uparrow & & & & \uparrow {\scriptstyle F(\alpha_{A,B,C})} \\
FA \bullet' (FB \bullet' FC) & \xrightarrow[\mathsf{id}_{FA} \bullet' m_{B,C}]{} & FA \bullet' F(B \bullet C) & \xrightarrow[m_{A,B \bullet C}]{} & F(A \bullet (B \bullet C))
\end{array}
$$

$$
\begin{array}{ccc}
FA \bullet' FB & \xrightarrow{m_{A,B}} & F(A \bullet B) \\
{\scriptstyle \gamma'_{A,B}} \downarrow & & \downarrow {\scriptstyle F(\gamma_{A,B})} \\
FB \bullet' FA & \xrightarrow[m_{B,A}]{} & F(B \bullet A)
\end{array}
$$

However in our particular case, assuming that ! is a symmetric monoidal (endo)functor means that ! comes equipped with a natural transformation

$$\mathsf{m}_{A,B}: !A \otimes !B \to !(A \otimes B)$$

and a morphism

$$\mathsf{m}_I: I \to !I$$

(where m_I is just the nullary version of the natural transformation.) The diagrams given in the above definition become the following:

$$
\begin{array}{ccc}
!I \otimes !A & \xrightarrow{\mathsf{m}_{I,A}} & !(I \otimes A) \\
{\scriptstyle \mathsf{m}_I \otimes \mathsf{id}_{!A}} \uparrow & & \downarrow {\scriptstyle !(\lambda_A)} \\
I \otimes !A & \xrightarrow[\lambda_{!A}]{} & !A
\end{array}
\qquad
\begin{array}{ccc}
!A \otimes !I & \xrightarrow{\mathsf{m}_{A,I}} & !(A \otimes I) \\
{\scriptstyle \mathsf{id}_{!A} \otimes \mathsf{m}_I} \uparrow & & \downarrow {\scriptstyle !(\rho_A)} \\
!A \otimes I & \xrightarrow[\rho_{!A}]{} & !A
\end{array}
$$

$$
\begin{array}{ccccc}
(!A \otimes !B) \otimes !C & \xrightarrow{\mathsf{m}_{A,B} \otimes \mathsf{id}_{!C}} & !(A \otimes B) \otimes !C & \xrightarrow{\mathsf{m}_{A \otimes B, C}} & !((A \otimes B) \otimes C) \\
{\scriptstyle \alpha_{!A,!B,!C}} \uparrow & & & & \uparrow {\scriptstyle !(\alpha_{A,B,C})} \\
!A \otimes (!B \otimes !C) & \xrightarrow[\mathsf{id}_{!A} \otimes \mathsf{m}_{B,C}]{} & !A \otimes !(B \otimes C) & \xrightarrow[\mathsf{m}_{A,B \otimes C}]{} & !(A \otimes (B \otimes C))
\end{array}
$$

$$\begin{array}{ccc} !A\otimes !B & \xrightarrow{\mathsf{m}_{A,B}} & !(A\otimes B) \\ {\scriptstyle \gamma_{A,B}}\downarrow & & \downarrow{\scriptstyle !(\gamma_{A,B})} \\ !B\otimes !A & \xrightarrow[\mathsf{m}_{B,A}]{} & !(B\otimes A) \end{array}$$

We have appropriate candidates for these monoidal morphisms in the interpretation of the proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{!A \vdash A}\,\textit{Dereliction} \qquad \dfrac{B \vdash B}{!B \vdash B}\,\textit{Dereliction}}{!A, !B \vdash A \otimes B}\,(\otimes_{\mathcal{R}})}{!A, !B \vdash !(A \otimes B)}\,\textit{Promotion}}{!A \otimes !B \vdash !(A \otimes B)}\,(\otimes_{\mathcal{L}}),$$

and

$$\dfrac{\dfrac{\vdash I}{\vdash !I}\,\textit{Promotion}}{I \vdash !I}\,(I_{\mathcal{L}}).$$

There are some extra notions of symmetric monoidal functors, depending on any extra properties of the morphisms. Let us give some examples.

Definition 26. A symmetric monoidal functor, $(F, m_{A,B}, m_{1'})\colon \mathbb{C} \to \mathbb{C}'$, is said to be

1. *Strict* if $m_{A,B}$ and $m_{1'}$ are identities.
2. *Strong* if $m_{A,B}$ and $m_{1'}$ are natural isomorphisms.

The equation in context for *Dereliction* gives us that

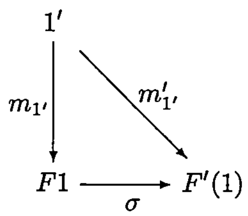
$$\begin{array}{ccccc} !A_1 & \xrightarrow{\mathsf{id}_{!A_1}} & !A_1 & \xrightarrow{(\epsilon_A; f)^*} & !B \\ & & & {\scriptstyle \epsilon_A; f}\searrow & \downarrow{\scriptstyle \epsilon_B} \\ & & & & B \end{array}$$

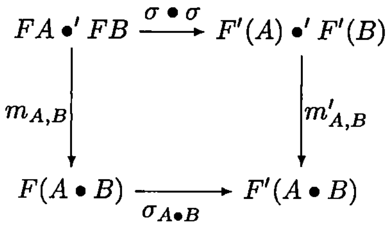
commutes, or in other words

$$\begin{array}{ccc} !A & \xrightarrow{!f} & !B \\ {\scriptstyle \epsilon_A}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

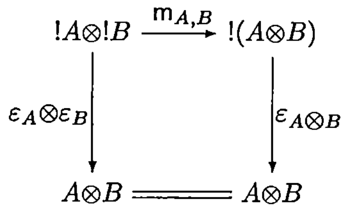
commutes. Given that we have made the assumption that ! is a (symmetric) monoidal functor, this diagram suggests that ϵ is a *monoidal natural transformation.* We shall make this assumption and write ε for the monoidal natural transformation $\varepsilon\colon ! \dot{\to} Id$. We shall repeat the definition of a monoidal natural transformation, which, again, is due to Eilenberg and Kelly.

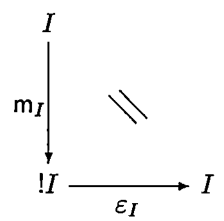
Definition 27. (Eilenberg and Kelly) A *monoidal natural transformation* between two monoidal functors $F, F'\colon \mathbb{C} \to \mathbb{C}'$ is a natural transformation $\sigma\colon F \dot{\to} F'$ which satisfies the following diagrams:





Again in our setup this says that the natural transformation $\varepsilon\colon ! \dot{\to} Id$ satisfies the following diagrams.



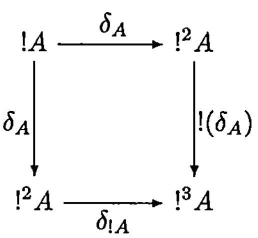


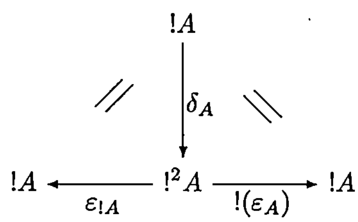
We have seen that from an identity morphism $\mathsf{id}_{!A}\colon !A \to !A$ we can form the canonical morphism $\delta = (\mathsf{id}_{!A})^*\colon !A \to !!A$. We know little about δ, other that given by equality 4.29, namely

$$\delta_A; \varepsilon_{!A} \quad = \quad \mathsf{id}_{!A}.$$

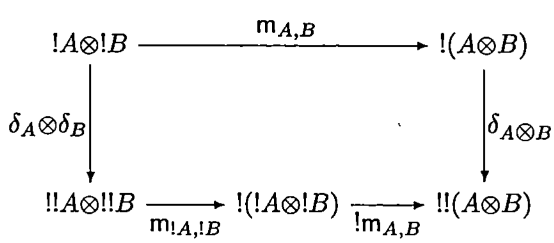
This equation is one of the three for a comonad. First let us repeat the definition of a comonad.

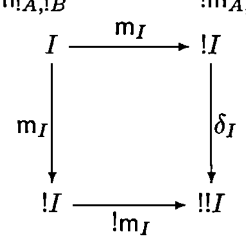
Definition 28. A *comonad* on a category $\mathbb{C}$ is a triple $(!, \varepsilon, \delta)$, where $!\colon \mathbb{C} \to \mathbb{C}$ is an endofunctor, and $\varepsilon\colon ! \dot{\to} Id$ and $\delta\colon !^2 \dot{\to} !$ are natural transformations, which make the following diagrams commute:





We shall assume that $(!, \varepsilon, \delta)$ form a comonad. In addition since $!$ and ε are monoidal, we shall assume that δ is also a *monoidal* natural transformation, and, hence, we have a (symmetric) *monoidal* comonad. The assumption that δ is a monoidal natural transformation amounts to requiring that the following diagrams commute.





We are now in a position to make a refined definition of the interpretation of the *Promotion* rule

$$\begin{aligned}
&[\![\Gamma_1,\ldots,\Gamma_n \triangleright \mathsf{promote}\ M_1,\ldots,M_n\ \mathsf{for}\ x_1,\ldots,x_n\ \mathsf{in}\ N{:}\,!B]\!]\\
&\stackrel{\text{def}}{=}\ \mathsf{Split_n}(\Gamma_1,\ldots,\Gamma_n);[\![\Gamma_1 \triangleright M_1{:}\,!A_1]\!]\otimes\ldots\otimes[\![\Gamma_n \triangleright M_n{:}\,!A_n]\!];\\
&\quad \delta\otimes\ldots\otimes\delta;\mathsf{m}_{!A_1,\ldots,!A_n};!([\![x_1,\ldots,x_n \triangleright N{:}\,!B]\!]).
\end{aligned} \tag{4.31}$$

As a slight aside, we can now reconsider the question of uniqueness for the factorization of the *Dereliction* rule (Rule 4.29) in the light of our new assumption of a comonad. We derived that the uniqueness assumption implied the equality $\varepsilon_{!A} = !\varepsilon_A$. We could then deduce the following

$$\begin{array}{rcll}
!\varepsilon_A;\delta_A &=& \delta_{!A};!!\varepsilon_A & \text{Naturality of } \delta\\
\varepsilon_{!A};\delta_A &=& \delta_{!A};!!\varepsilon_A & \text{Rule 4.30}\\
\varepsilon_{!A};\delta_A &=& \delta_{!A};!\varepsilon_{!A} & \text{Rule 4.30}\\
\varepsilon_{!A};\delta_A &=& \mathsf{id}_{!A} & \text{Def of a comonad.}
\end{array}$$

Since $\delta_A;\varepsilon_{!A} = \mathsf{id}_{!A}$ by definition of a comonad, we can see that the uniqueness assumption would have lead to the model collapsing to the extent that $!A \cong !!A$, or, in other words, that the comonad was *idempotent*. It should be made clear that our model does *not* have an idempotent comonad.

Finally one of the comonad equalities suggests a term equality as follows

$$\begin{array}{rcl}
\delta;!(\varepsilon) &=& \mathsf{id}_{!A}\\
\mathsf{id}_{!A};\delta;!(\mathsf{id}_{!A};\varepsilon) &=& \mathsf{id}_{!A}\\
[\![x{:}\,!A \triangleright x{:}\,!A]\!];\delta;!([\![y{:}\,!A \triangleright \mathsf{derelict}(y){:}\,A]\!]) &=& [\![x{:}\,!A \triangleright x{:}\,!A]\!]\\
[\![x{:}\,!A \triangleright \mathsf{promote}\ x\ \mathsf{for}\ y\ \mathsf{in}\ \mathsf{derelict}(y){:}\,!A]\!] &=& [\![x{:}\,!A \triangleright x{:}\,!A]\!].
\end{array}$$

Or, in other words, the (new) equation in context

$$\frac{}{x{:}\,!A \triangleright \mathsf{promote}\ x\ \mathsf{for}\ y\ \mathsf{in}\ \mathsf{derelict}(y) = x{:}\,!A}\ \textit{Comonad.}$$

The rule for *Weakening* is of the form

$$\frac{\Gamma \triangleright M{:}\,!A \qquad \Delta \triangleright N{:}\,B}{\Gamma,\Delta \triangleright \mathsf{discard}\ M\ \mathsf{in}\ N{:}\,B}\ \textit{Weakening.}$$

To interpret this rule we need a natural transformation with components

$$\Phi_{\Gamma,\Delta}{:}\,\mathbb{C}(\Gamma,!A)\times\mathbb{C}(\Delta,B)\to\mathbb{C}((\Gamma,\Delta),B).$$

Given morphisms $e{:}\,\Gamma \to !A$, $f{:}\,\Delta \to B$, $c{:}\,\Gamma' \to \Gamma$ and $d{:}\,\Delta' \to \Delta$, naturality gives the equation

$$\mathsf{Split}(\Gamma',\Delta');c\otimes d;\mathsf{Join}(\Gamma,\Delta);\Phi_{\Gamma,\Delta}(e,f) \quad=\quad \Phi_{\Gamma',\Delta'}((c;e),(d;f)). \tag{4.32}$$

In particular if we take e to be $\mathsf{id}_{!A}$, f to be id_B, c to be a morphism $m{:}\,\Gamma \to !A$ and d to be a morphism $n{:}\,\Delta \to B$ then by naturality we have

$$\mathsf{Split}(\Gamma,\Delta);m\otimes n;\Phi_{!A,B}(\mathsf{id}_{!A},\mathsf{id}_B) \quad=\quad \Phi_{\Gamma,\Delta}(m,n).$$

We shall make the simplifying assumption that the natural transformation Φ is also natural in B. Given morphisms $e{:}\,\Gamma \to !A$, $f{:}\,\Delta \to B$ and $g{:}\,B\otimes\Theta \to C$, this gives

$$\mathsf{Split}((\Gamma,\Delta),\Theta);\Phi(e,f)\otimes\mathsf{id}_\Theta;g \quad=\quad \Phi(e,(\mathsf{Split}(\Delta,\Theta);f\otimes\mathsf{id}_\Theta;g)). \tag{4.33}$$

At the level of terms this gives the equation in context

$$\frac{\Gamma \triangleright N\colon !A \qquad \Delta \triangleright M\colon B \qquad x\colon B, \Theta \triangleright P\colon C}{\Gamma, \Delta, \Theta \triangleright P[x := \mathsf{discard}\ M\ \mathsf{in}\ N] = \mathsf{discard}\ M\ \mathsf{in}\ P[x := N]\colon C}\ \mathit{Weakening}_{nat}.$$

Let us consider the second naturality equation (4.33) with the morphisms $e = \lambda_{!A}\colon I\otimes !A \to !A$, $f = \mathsf{id}_{I\otimes B}$ and $g = \lambda_B\colon I\otimes B \to B$, then we get the equality

$$\Phi(\lambda_{!A}, \lambda_B) \quad = \quad \Phi(\lambda_{!A}, \mathsf{id}_{I\otimes B}); \lambda_B. \tag{4.34}$$

Again, let us consider the second naturality equation (4.33) but with the morphisms $e = \lambda_{!A}\colon I\otimes !A \to !A$, $f = \mathsf{id}_I$ and $g = \mathsf{id}_{I\otimes B}$, then we get the equality

$$\alpha^{-1}_{I\otimes !A,I,B}; \Phi(\lambda_{!A}, \mathsf{id}_{I\otimes B}) \quad = \quad \Phi(\lambda_{!A}, \mathsf{id}_I)\otimes \mathsf{id}_B; \mathsf{id}_{I\otimes B}. \tag{4.35}$$

Now if we take the first naturality equation (4.32) with the morphisms $e = \lambda_{!A}$, $c = \lambda^{-1}_{!A}$, $f = \lambda_B$ and $d = \lambda^{-1}_B$ we get the equality

$$\begin{array}{rcll}
\Phi((\lambda^{-1}_{!A}; \lambda_{!A}), (\lambda^{-1}_B; \lambda_B)) & = & \lambda^{-1}_{!A}\otimes\lambda^{-1}_B; \alpha; \Phi(\lambda_{!A}, \lambda_B) & \\
\Phi(\mathsf{id}_{!A}, \mathsf{id}_B) & = & \lambda^{-1}_{!A}\otimes\lambda^{-1}_B; \alpha; \Phi(\lambda_{!A}, \lambda_B) & \\
 & = & \lambda^{-1}_{!A}\otimes\lambda^{-1}_B; \alpha; \Phi(\lambda_{!A}, \mathsf{id}_{I\otimes B}); \lambda_B & \text{Rule 4.34} \\
 & = & \lambda^{-1}_{!A}\otimes\lambda^{-1}_B; \alpha; \Phi(\lambda_{!A}, \mathsf{id}_I)\otimes\mathsf{id}_B; \lambda_B & \text{Rule 4.35} \\
 & = & \lambda^{-1}_{!A}\otimes\mathsf{id}; \mathsf{id}\otimes\lambda^{-1}_B; \alpha; \Phi(\lambda, \mathsf{id})\otimes\mathsf{id}; \lambda & \\
 & = & \lambda^{-1}_{!A}\otimes\mathsf{id}; \rho^{-1}_{I\otimes !A}\otimes\mathsf{id}; \Phi(\lambda_{!A}, \mathsf{id}_I)\otimes\mathsf{id}_B; \lambda_B & \text{Def of a SMC} \\
 & = & (\lambda^{-1}; \rho^{-1}; \Phi(\lambda_{!A}, \mathsf{id}_I))\otimes\mathsf{id}_B; \lambda_B. &
\end{array}$$

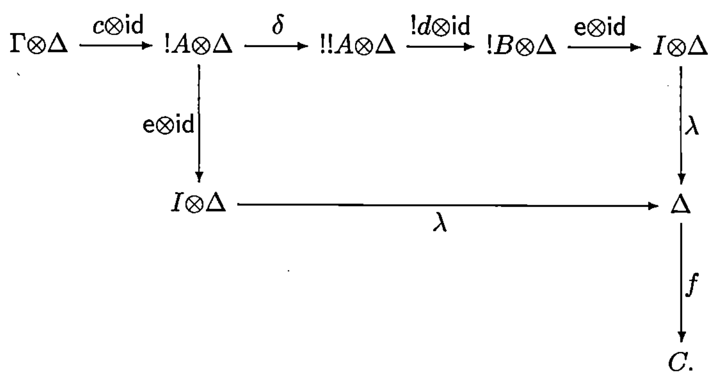
The operation $\lambda^{-1}; \rho^{-1}; \Phi(\lambda_{!A}, \mathsf{id}_I)$ we shall denote as $\mathsf{e}\colon !A \to I$. Thus we can express the morphism $\Phi(\mathsf{id}_{!A}, \mathsf{id}_B)$ in terms of this simpler operation. We can then make the definition

$$[\![\Gamma, \Delta \triangleright \mathsf{discard}\ M\ \mathsf{in}\ N\colon B]\!] \stackrel{\text{def}}{=} \mathsf{Split}(\Gamma, \Delta); [\![\Gamma \triangleright M\colon !A]\!]\otimes[\![\Delta \triangleright N\colon B]\!]; \mathsf{e}\otimes\mathsf{id}_B; \lambda_B. \tag{4.36}$$

We have the following equation in context for *Weakening*

$$\frac{\begin{array}{l}\Gamma_1 \triangleright M_1\colon !A_1 \\ \Gamma_n \triangleright M_n\colon !A_n\end{array} \qquad x_1\colon !A_1, \ldots, x_n\colon !A_n \triangleright N\colon B \qquad \Delta \triangleright P\colon C}{\begin{array}{ll}\Gamma_1, \ldots, \Gamma_n, \Delta \triangleright & \mathsf{discard}\ (\mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N)\ \mathsf{in}\ P \\ & = \mathsf{discard}\ M_1, \ldots, M_n\ \mathsf{in}\ P\colon C\end{array}}\ \mathit{Disc}_{Eq}.$$

To simplify the presentation, let us consider the case when $n = 1$. If we take morphisms $c\colon \Gamma \to !A$, $d\colon !A \to B$ and $f\colon \Delta \to C$, this rule amounts to the diagram



The rule for *Contraction* is of the form

$$\frac{\Delta \rhd M\colon !A \qquad \Gamma, x\colon !A, y\colon !A \rhd N\colon B}{\Gamma, \Delta \rhd \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N\colon B}\ \textit{Contraction.}$$

To interpret this rule we need a natural transformation with components

$$\Phi_{\Gamma,\Delta}\colon \mathbb{C}(\Delta, !A) \times \mathbb{C}(\Gamma\otimes!A\otimes!A, B) \to \mathbb{C}((\Gamma,\Delta), !B).$$

Given morphisms $e\colon \Delta \to !A$, $f\colon \Gamma\otimes!A\otimes!A \to B$, $c\colon \Gamma' \to \Gamma$ and $d\colon \Delta' \to \Delta$, naturality gives the equation

$$\mathsf{Split}(\Gamma',\Delta'); c\otimes d; \mathsf{Join}(\Gamma,\Delta); \Phi_{\Delta,\Gamma}(e,f) \quad = \quad \Phi_{\Delta',\Gamma'}((d;e),(c\otimes \mathsf{id}_{!A}\otimes \mathsf{id}_{!A}; f)).$$

In particular if we take e to be $\mathsf{id}_{!A}$, f to be a morphism $m\colon \Gamma\otimes!A\otimes!A \to B$, c to be id_Γ and d to be a morphism $m\colon \Delta \to !A$, then by naturality we have

$$\mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma\otimes m; \Phi_{!A,\Gamma}(\mathsf{id}_{!A}, n) \quad = \quad \Phi_{\Gamma,\Delta}(m,n).$$

Thus $\Phi_{\Gamma,\Delta}(m,n)$ can be expressed as the composite $\mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma\otimes m; \Psi_\Gamma(n)$, where Ψ is a natural transformation with components

$$\Psi_\Gamma\colon \mathbb{C}(\Gamma\otimes!A\otimes!A, B) \to \mathbb{C}(\Gamma\otimes!A, B).$$

We shall make the simplifying assumption that Φ is natural in B. Given the definition above, this implies that Ψ should be natural in B. Given morphisms $e\colon \Gamma\otimes!A\otimes!A \to B$ and $f\colon \Theta\otimes B \to C$, this gives the equation

$$\mathsf{Split}(\Theta,(\Gamma,!A)); \mathsf{id}_\Theta\otimes\Psi(e); f \quad = \quad \Psi(\mathsf{Split}(\Theta,\Gamma)\otimes\mathsf{id}_{!A}\otimes\mathsf{id}_{!A}; \alpha^{-1}\otimes\mathsf{id}_{!A}; \alpha^{-1}; \mathsf{id}_\Theta\otimes e; f). \quad (4.37)$$

At the level of terms this gives the equation in context

$$\frac{\Delta \rhd M\colon !A \qquad \Gamma, x\colon !A, y\colon !A \rhd N\colon B \qquad \Theta, z\colon B \rhd P\colon C}{\Theta, \Gamma, \Delta \rhd P[z := \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N] = \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ P[z := N]\colon C}\ \textit{Contraction}_{nat}.$$

If we take the naturality equation 4.37 with the morphisms $e = \mathsf{id}_{!A}\otimes\mathsf{id}_{!A}$ and $f = m\colon \Gamma\otimes!A\otimes!A \to B$ we get the equality

$$\Psi(f) \quad = \quad \mathsf{id}_\Gamma\otimes\Psi(\mathsf{id}_{!A}\otimes\mathsf{id}_{!A}); f.$$

If we define $\mathsf{d} \stackrel{\text{def}}{=} \Psi(\mathsf{id}_{!A}\otimes\mathsf{id}_{!A})$, then we can make the definition

$$[\![\Gamma, \Delta \rhd \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N\colon B]\!] \stackrel{\text{def}}{=}$$
$$\mathsf{Split}(\Gamma,\Delta); \mathsf{id}_\Gamma\otimes[\![\Delta \rhd M\colon !A]\!]; \mathsf{id}_\Gamma\otimes\mathsf{d}; [\![\Gamma, x\colon !A, y\colon !A \rhd N\colon B]\!]. \quad (4.38)$$

We have the following equation in context for *Contraction*

$$\frac{\begin{array}{l}\Gamma_1 \rhd M_1\colon !A_1 \\ \Gamma_n \rhd M_n\colon !A_n\end{array} \qquad x_1\colon !A_1,\ldots,x_n\colon !A_n \rhd N\colon B \qquad \Delta, x\colon !B, y\colon !B \rhd P\colon C}{\begin{array}{rl}\Gamma_1,\ldots,\Gamma_n,\Delta \rhd & \mathsf{copy}\ (\mathsf{promote}\ M_1,\ldots,M_n\ \mathsf{for}\ x_1,\ldots,x_n\ \mathsf{in}\ N)\ \mathsf{as}\ x,y\ \mathsf{in}\ P \\ = & \mathsf{copy}\ M_1,\ldots,M_n\ \mathsf{as}\ (x_1',\ldots,x_n'),(x_1'',\ldots,x_n'')\ \mathsf{in} \\ & P[x := \mathsf{promote}\ x_1',\ldots,x_n'\ \mathsf{for}\ x_1,\ldots,x_n\ \mathsf{in}\ N, \\ & \quad y := \mathsf{promote}\ x_1'',\ldots,x_n''\ \mathsf{for}\ x_1,\ldots,x_n\ \mathsf{in}\ N]\colon C\end{array}}\ \textit{Copy}_{Eq}.$$

To simplify the presentation as earlier, let us consider the case when $n = 1$. If we take morphisms $m: \Delta \to !A$, $n: !A \to B$ and $p: \Gamma \otimes !B \otimes !B \to C$, this rule amounts to the diagram

$$\begin{array}{ccccccc}
\Gamma\otimes\Delta & \xrightarrow{\mathsf{id}\otimes m} & \Gamma\otimes !A & \xrightarrow{\mathsf{id}\otimes\delta} & \Gamma\otimes !!A & \xrightarrow{\mathsf{id}\otimes !n} & \Gamma\otimes !B \\
 & & \downarrow{\scriptstyle \mathsf{id}\otimes\mathsf{d}} & & & & \downarrow{\scriptstyle \mathsf{id}\otimes\mathsf{d}} \\
 & & \Gamma\otimes !A\otimes !A & \xrightarrow[\mathsf{id}\otimes\delta\otimes\delta]{} & \Gamma\otimes !!A\otimes !!A & \xrightarrow[\mathsf{id}\otimes !n\otimes !n]{} & \Gamma\otimes !B\otimes !B \\
 & & & & & & \downarrow{\scriptstyle p} \\
 & & & & & & C.
\end{array}$$

Let us review what we know so far about the rules of *Weakening* and *Contraction.* We have established that they are modelled with the use of morphisms $\mathsf{e}: !A \to I$ and $\mathsf{d}: !A \to !A\otimes !A$, and that the following diagrams commute for a morphism $f: !A \to B$

$$\begin{array}{ccccc}
!A & \xrightarrow{\delta} & !!A & \xrightarrow{!f} & !B \\
\downarrow{\scriptstyle \mathsf{e}_A} & & & & \downarrow{\scriptstyle \mathsf{e}_B} \\
I & = & = & = & I
\end{array}$$

and

$$\begin{array}{ccccc}
!A & \xrightarrow{\delta} & !!A & \xrightarrow{!f} & !B \\
\downarrow{\scriptstyle \mathsf{d}_A} & & & & \downarrow{\scriptstyle \mathsf{d}_A} \\
!A\otimes !A & \xrightarrow[\delta\otimes\delta]{} & !!A\otimes !!A & \xrightarrow[!f\otimes !f]{} & !B\otimes !B.
\end{array}$$

We can see that the morphisms e and d act upon morphisms of the form $!f$ (for some f) and δ. This suggests some other categorical properties of the morphisms. Firstly, we should expect that e and d are *natural transformations* and hence both

$$\begin{array}{ccc}
!A & \xrightarrow{!f} & !B \\
\downarrow{\scriptstyle \mathsf{e}_A} & & \downarrow{\scriptstyle \mathsf{e}_B} \\
I & = & I
\end{array}$$

and

$$\begin{array}{ccc}
!A & \xrightarrow{!f} & !B \\
\downarrow{\scriptstyle \mathsf{d}_A} & & \downarrow{\scriptstyle \mathsf{d}_B} \\
!A\otimes !A & \xrightarrow[!f\otimes !f]{} & !B\otimes !B
\end{array}$$

should commute for any morphism $f: A \to B$.We might also like to assume that d and e are *monoidal* natural transformations. First we need the following proposition.

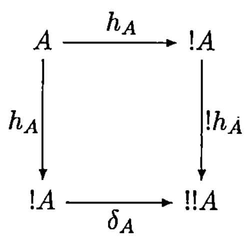
Proposition 6. Given a monoidal functor, $(!, \mathsf{m}_{A,B}, \mathsf{m}_I)$, on a SMC, $\mathbb{C}$, then both the functors $!\otimes!: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $I: \mathbb{C} \to I$[7] are monoidal.

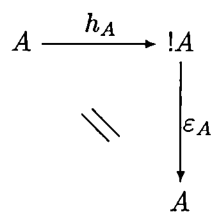
Proof. Trivial, by construction. ■

We shall make the simplifying assumption that d and e are monoidal natural transformations and we shall spell out the categorical consequences in the next section.

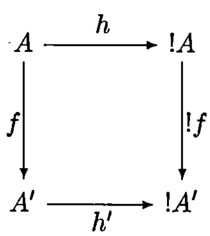
Before considering the categorical import of d and e acting upon δ, let us review some standard categorical structures.

Definition 29. Given a comonad, $(!, \varepsilon, \delta)$, on a category $\mathbb{C}$, a *coalgebra* is a pair $(A, h_A: A \to !A)$ where A is an object of $\mathbb{C}$ and h_A is a morphism in $\mathbb{C}$ (called the 'structure map' of the coalgebra) which makes the following diagrams commute.





A *coalgebra morphism* between two coalgebras (A, h) and (A', h') is a morphism $f: A \to A'$ in $\mathbb{C}$ which makes the following diagram commute.

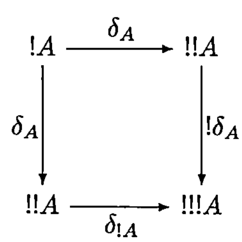


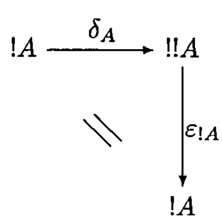
Definition 30. Given a comonad, $(!, \varepsilon, \delta)$, on a category $\mathbb{C}$, we can form the *category of coalgebras*, $\mathbb{C}^!$, with objects the coalgebras and morphisms the coalgebra morphisms in $\mathbb{C}$.

The opposite notion of this construction (i.e. that generated by a monad) is often referred to as the "Eilenberg-Moore category" [53, Page 136]. Given this definition we can identify some useful coalgebras which exist in a SMC with a (symmetric) monoidal comonad.

Proposition 7. Given a comonad, $(!, \varepsilon, \delta)$, on a SMC $\mathbb{C}$, then $(!A, \delta_A: !A \to !!A)$ is a coalgebra.

Proof. We require the following diagrams to commute.





Both these diagrams commute from the definition of a comonad. ■

[7] Where I is the constant functor which maps the category $\mathbb{C}$ to the one object category.

Proposition 8. Given a monoidal comonad, $(!, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_I)$, on a SMC $\mathbb{C}$, then $(I, \mathsf{m}_I \colon I \to !I)$ is a coalgebra.

Proof. We require the following diagrams to commute.

$$\begin{array}{ccc} I & \xrightarrow{\mathsf{m}_I} & !I \\ {\scriptstyle \mathsf{m}_I}\big\downarrow & & \big\downarrow{\scriptstyle !\mathsf{m}_I} \\ !I & \xrightarrow[\delta_I]{} & !!I \end{array} \qquad\qquad \begin{array}{ccc} I & \xrightarrow{\mathsf{m}_I} & !I \\ & \diagdown\!\diagdown & \big\downarrow{\scriptstyle \varepsilon_I} \\ & & I \end{array}$$

The left hand diagram commutes since δ is a monoidal natural transformation, and the right hand diagram commutes because ε is a monoidal natural transformation. ∎

Proposition 9. Given a monoidal comonad, $(!, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_I)$, on a SMC $\mathbb{C}$, then $(!A \otimes !A, (\delta_A \otimes \delta_A ; \mathsf{m}_{!A,!A}))$ is a coalgebra.

Proof. We require the following diagrams to commute.

$$\begin{array}{ccccc} !A\otimes !A & \xrightarrow{\delta\otimes\delta} & !!A\otimes !!A & \xrightarrow{\mathsf{m}_{!A,!A}} & !(!A\otimes !A) \\ {\scriptstyle \delta\otimes\delta}\big\downarrow & & \big\downarrow{\scriptstyle !\delta\otimes !\delta} & & \big\downarrow{\scriptstyle !(\delta\otimes\delta)} \\ !!A\otimes !!A & \xrightarrow{\delta_{!A}\otimes\delta_{!A}} & !!!A\otimes !!!A & \xrightarrow{\mathsf{m}} & !(!!A\otimes !!A) \\ {\scriptstyle \mathsf{m}}\big\downarrow & & & & \big\downarrow{\scriptstyle !\mathsf{m}} \\ !(!A\otimes !A) & & \xrightarrow[\delta]{} & & !!(!A\otimes !A) \end{array}$$

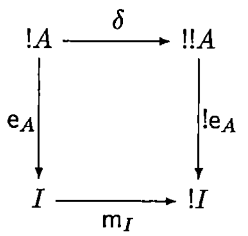
$$\begin{array}{ccccc} !A\otimes !A & \xrightarrow{\delta\otimes\delta} & !!A\otimes !!A & \xrightarrow{\mathsf{m}} & !(!A\otimes !A) \\ & \diagdown\!\diagdown & \big\downarrow{\scriptstyle \varepsilon\otimes\varepsilon} & & \big\downarrow{\scriptstyle \varepsilon} \\ & & !A\otimes !A & = & !A\otimes !A \end{array}$$

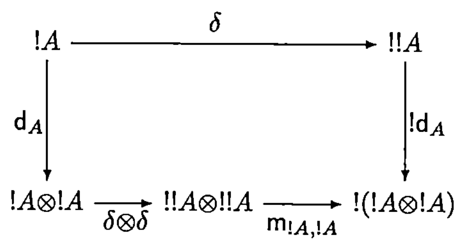
In the top diagram the upper left square commutes by definition of a comonad, the upper right square commutes since m is a natural transformation and the lower square commutes since δ is a monoidal natural transformation. In the lower diagram, the left hand triangle holds by the definition of a comonad, and the square commutes since ε is a monoidal natural transformation. ∎

Proposition 10. Given a comonad, $(!, \delta, \varepsilon)$, on a category $\mathbb{C}$, then given a coalgebra $(A, h_A \colon A \to !A)$, the structure map, h_A, is also a coalgebra morphism between the coalgebras (A, h_A) and $(!A, \delta)$.

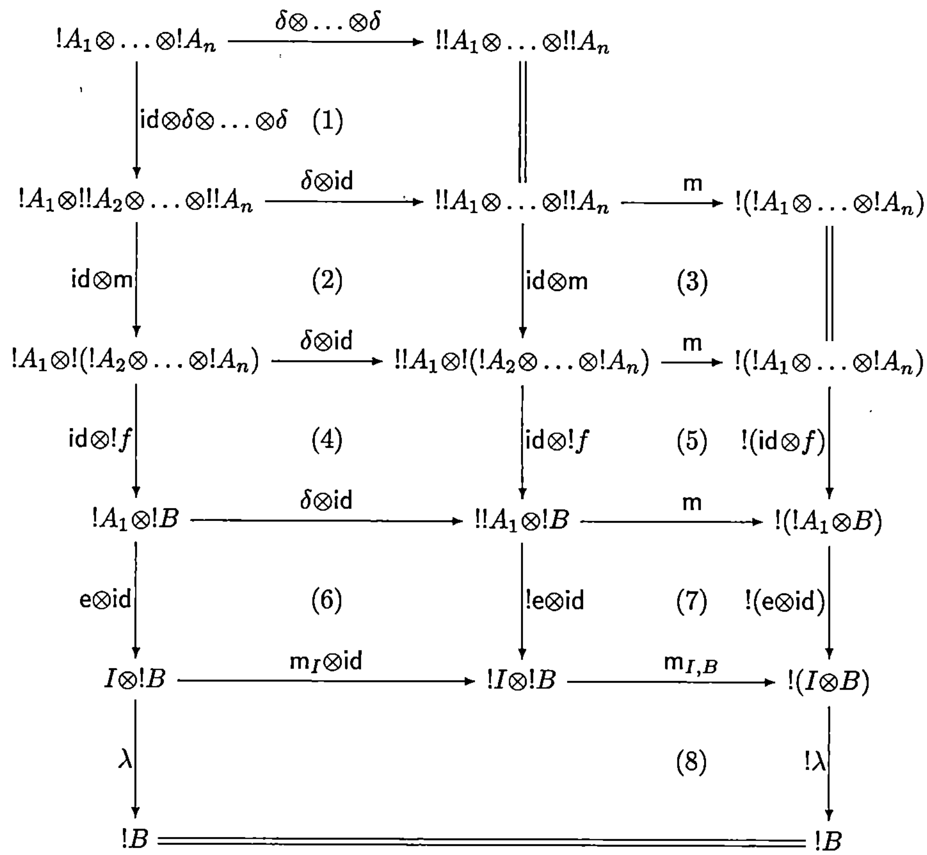
Proof. By definition of h_A being a structure map. ∎

Returning to the question of the relationship between d, e and δ, we might now expect that e and d are *coalgebra morphisms*. This assumption amounts to requiring the following diagrams to commute.





These equalities lead to new equations in context. The left hand diagram above gives the force of the following diagram commuting.



Squares (1), (2) and (4) commute trivially. Squares (3) and (8) commute by definition of m. Squares (5) and (7) commutes by naturality of m. Square (6) commutes by our new assumption that e is a coalgebra morphism.

This diagram amounts to the following (new) equation in context.

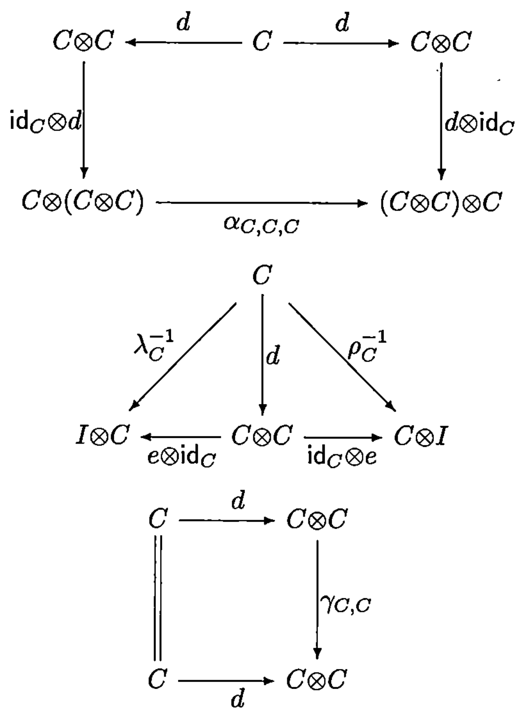
$$\frac{\begin{array}{ll}\Gamma_1 \rhd M_1\colon !A_1 & \\ \Gamma_n \rhd M_n\colon !A_n & x_2\colon !A_2,\ldots,x_n\colon !A_n \rhd N\colon B\end{array}}{\begin{array}{rl}\Gamma_1,\ldots,\Gamma_n \rhd & \mathsf{promote}\ (M_1,\ldots,M_n)\ \mathsf{for}\ (x_1,\ldots,x_n)\ \mathsf{in\ discard}\ x_1\ \mathsf{in}\ N \\ & = \mathsf{discard}\ M_1\ \mathsf{in\ promote}\ (M_2,\ldots,M_n)\ \mathsf{for}\ (x_2,\ldots,x_n)\ \mathsf{in}\ N\colon !B\end{array}}\ \mathit{Coalgebra}_1$$

In a similar way (which we shall omit), assuming that d is a coalgebra morphism makes another large diagram commute. It amounts to the following new equation in context.

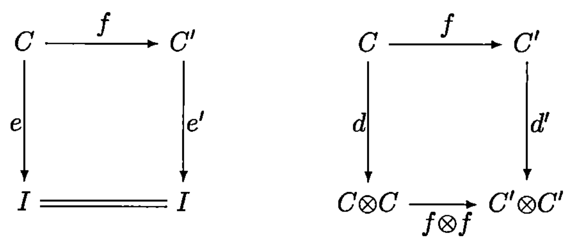
$$\frac{\begin{array}{ll}\Gamma_1 \rhd M_1\colon !A_1 & \\ \Gamma_n \rhd M_n\colon !A_n & y\colon !A_1, z\colon !A_1, x_2\colon !A_2,\ldots,x_n\colon !A_n \rhd N\colon B\end{array}}{\begin{array}{rll}\Gamma_1,\ldots,\Gamma_n \rhd & \multicolumn{2}{l}{\mathsf{promote}\ (M_1,\ldots,M_n)\ \mathsf{for}\ (x_1,\ldots,x_n)\ \mathsf{in\ copy}\ x_1\ \mathsf{as}\ y,z\ \mathsf{in}\ N} \\ & = & \mathsf{copy}\ M_1\ \mathsf{as}\ y',z'\ \mathsf{in} \\ & & \mathsf{promote}\ (y',z',M_2,\ldots,M_n)\ \mathsf{for}\ (y,z,x_2,\ldots,x_n)\ \mathsf{in}\ N\colon !B\end{array}}\ \mathit{Coalgebra}_2$$

We have a final piece of categorical structure which it seems useful to utilize. First we shall review some of the standard definitions.

Definition 31. Given a SMC, $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$, a *commutative comonoid in* $\mathbb{C}$ is a triple (C, d, e) where C is an object in $\mathbb{C}$ and $d: C \to C \otimes C$ and $e: C \to I$ are morphisms in $\mathbb{C}$ such that the following diagrams commute.

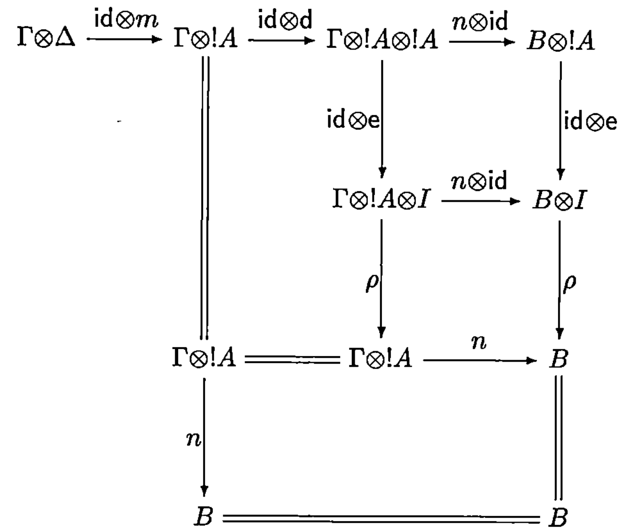


Definition 32. A *comonoid morphism* between two comonoids (C, d, e) and (C', d', e') is a morphism $f: C \to C'$ such that the following diagrams commute



Definition 33. Given a SMC, $\mathbb{C}$, the *category of commutative comonoids*, $\mathrm{coMon_c}(\mathbb{C})$, has as objects the commutative comonoids and morphisms the comonoid morphisms in $\mathbb{C}$.

It would seem appealing (from both a computational and categorical viewpoint) to assume that d and e form a commutative comonoid, $(!A, \mathsf{d}, \mathsf{e})$. Again this assumption provides some extra equations in context. For example, take morphisms $m: \Delta \to !A$ and $n: \Gamma \otimes !A \to B$, then the following diagram commutes.



The left hand square commutes by the assumption that d and e form a commutative comonoid. The upper right hand square commutes trivially and the middle right hand square commutes by naturality of ρ.

This diagram amounts to the following equation in context.

$$\frac{\Delta \triangleright M\colon !A \qquad \Gamma, y\colon !A \triangleright N\colon B}{\Gamma, \Delta \triangleright \mathsf{copy}\ M\ \mathsf{as}\ x, y\ \mathsf{in}\ \mathsf{discard}\ x\ \mathsf{in}\ N = N[y := M]\colon B}\ \mathit{Comonoid}_1$$

Similar reasoning using the other properties of a commutative comonoid gives three other equations in context.

$$\frac{\Delta \triangleright M\colon !A \qquad \Gamma, x\colon !A \triangleright N\colon B}{\Gamma, \Delta \triangleright \mathsf{copy}\ M\ \mathsf{as}\ x, y\ \mathsf{in}\ \mathsf{discard}\ x\ \mathsf{in}\ N = N[y := M]\colon B}\ \mathit{Comonoid}_2$$

$$\frac{\Delta \triangleright M\colon !A \qquad \Gamma, x\colon !A, y\colon !A, \triangleright N\colon B}{\Gamma, \Delta \triangleright \mathsf{copy}\ M\ \mathsf{as}\ x, y\ \mathsf{in}\ N = \mathsf{copy}\ M\ \mathsf{as}\ y, x\ \mathsf{in}\ N\colon B}\ \mathit{Comonoid}_3$$

$$\frac{\Delta \triangleright M\colon !A \qquad \Gamma, x\colon !A, y\colon !A, z\colon !A \triangleright N\colon B}{\begin{array}{ll}\Gamma, \Delta \triangleright & \mathsf{copy}\ M\ \mathsf{as}\ x, w\ \mathsf{in}\ \mathsf{copy}\ w\ \mathsf{as}\ y, z\ \mathsf{in}\ N \\ & = \mathsf{copy}\ M\ \mathsf{as}\ w, z\ \mathsf{in}\ \mathsf{copy}\ w\ \mathsf{as}\ x, y\ \mathsf{in}\ N\colon B\end{array}}\ \mathit{Comonoid}_4$$

In the next section we shall summarize the analysis given in this section and define a categorical model for **ILL**. As we have seen, the analysis so far has produced some new equations in context beyond those given by β-reduction. For completeness we shall repeat these new equations in context in Figures 4.3, 4.4 and 4.5.

Using these new equations in context we can now define what we consider to be a linear term calculus theory.

Definition 34. A *linear term calculus theory (LTC-theory)*, $\mathcal{T} = (\mathcal{L}, \mathcal{A})$ where

- $\mathcal{L}$ is a LTC-signature (Definition 24)
- $\mathcal{A}$ are the equations in context given in Figures 4.2, 4.3, 4.4 and 4.5.

$$\frac{\Gamma \rhd N: A \multimap B}{\Gamma \rhd \lambda x: A.Nx = N: A \multimap B} \multimap_\eta$$

$$\frac{\Delta \rhd M: A \otimes B \qquad \Gamma, z: A \otimes B \rhd N: C}{\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ N[z := x \otimes y] = N[z := M]: C} \otimes_\eta$$

$$\frac{\Delta \rhd M: I \qquad \Gamma, z: I \rhd N: A}{\Gamma, \Delta \rhd \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N[z := *] = N[z := M]: A} I_\eta$$

$$\frac{\Gamma \rhd M: A \& B}{\Gamma \rhd \langle \mathsf{fst}(M), \mathsf{snd}(M) \rangle = M: A \& B} \&_\eta$$

$$\frac{\Gamma \rhd M: A \oplus B}{\Gamma \rhd \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to \mathsf{inl}(x) \parallel \mathsf{inr}(y) \to \mathsf{inr}(y) = M: A \oplus B} \oplus_\eta$$

Figure 4.3: 'η' Equations in Context

4 The Model for Intuitionistic Linear Logic

In this section we sum up the analysis given in the previous sections by detailing the categorical model for **ILL**.[8] First we shall define the model and then consider some properties of the definition.

Definition 35. A *Linear category*, $\mathbb{C}$, consists of:

1. A SMCC, $\mathbb{C}$, with finite products and coproducts, together with:
2. A symmetric monoidal comonad $(!, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_I)$ such that
   (a) For every free !-coalgebra $(!A, \delta_A)$ there are two distinguished monoidal natural transformations with components $e_A: !A \to I$ and $d_A: !A \to !A \otimes !A$ which form a commutative comonoid and are coalgebra morphisms.
   (b) Whenever $f: (!A, \delta_A) \to (!B, \delta_B)$ is a coalgebra morphism between free coalgebras, then it is also a comonoid morphism.

Let us consider in detail the conditions in this definition. Requiring that $e_A: !A \to I$ is a monoidal natural transformation amounts to requiring that the following three diagrams commute, for any morphism $f: A \to B$.

$$\begin{array}{ccc} !A & \xrightarrow{e_A} & I \\ {\scriptstyle !f}\downarrow & & \parallel \\ !B & \xrightarrow[e_B]{} & I \end{array}$$

[8]The multiplicative, exponential part of this model is originally due to de Paiva and Hyland and appeared in a preliminary form in [15]. That presented here is a slight simplification of their model and is also extended to model the additives.

$$\frac{\Delta \rhd M\colon A\otimes B \qquad \Gamma, x\colon A, y\colon B \rhd N\colon C \qquad \Theta, z\colon C \rhd P\colon D}{\Theta,\Gamma,\Delta \rhd P[z := \mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ N] = \mathsf{let}\ M\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ P[z := N]\colon C}\ \otimes_{nat}$$

$$\frac{\Gamma \rhd N\colon A \qquad \Delta \rhd M\colon I \qquad \Theta, z\colon A \rhd P\colon B}{\Theta,\Gamma,\Delta \rhd P[z := \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ N] = \mathsf{let}\ M\ \mathsf{be}\ *\ \mathsf{in}\ P[z := N]\colon B}\ I_{nat}$$

$$\frac{\Delta \rhd M\colon A\oplus B \quad \Gamma, x\colon A \rhd N\colon C \quad \Gamma, y\colon B \rhd P\colon C \quad \Theta, z\colon C \rhd Q\colon D}{\begin{array}{lll}\Theta,\Gamma,\Delta & \rhd & Q[z := \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to N \parallel \mathsf{inr}(y) \to P] \\ & & = \mathsf{case}\ M\ \mathsf{of}\ \mathsf{inl}(x) \to (Q[z := N]) \parallel \mathsf{inr}(y) \to (Q[z := P])\colon D\end{array}}\ \oplus_{nat}$$

$$\frac{\Gamma \rhd N\colon !A \qquad \Delta \rhd M\colon B \qquad x\colon B, \Theta \rhd P\colon C}{\Gamma,\Delta,\Theta \rhd P[x := \mathsf{discard}\ M\ \mathsf{in}\ N] = \mathsf{discard}\ M\ \mathsf{in}\ P[x := N]\colon C}\ \mathit{Weakening}_{nat}$$

$$\frac{\Delta \rhd M\colon !A \qquad \Gamma, x\colon !A, y\colon !A \rhd N\colon B \qquad \Theta, z\colon B \rhd P\colon C}{\Theta,\Gamma,\Delta \rhd P[z := \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N] = \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ P[z := N]\colon C}\ \mathit{Contraction}_{nat}$$

Figure 4.4: 'Naturality' Equations in Context

$$\begin{array}{ccc} I & & \\ {\scriptstyle \mathsf{m}_I}\downarrow & \diagdown\!\diagdown & \\ !I & \xrightarrow[e_I]{} & I \end{array} \qquad\qquad \begin{array}{ccc} !A\otimes !B & \xrightarrow{e_A\otimes e_B} & I\otimes I \\ {\scriptstyle \mathsf{m}_{A,B}}\downarrow & & \downarrow{\scriptstyle \lambda_I} \\ !(A\otimes B) & \xrightarrow[e_{A\otimes B}]{} & I \end{array}$$

Requiring that $d_A\colon !A \to !A\otimes !A$ is a monoidal natural transformation amounts to requiring that the following three diagrams commute, for all $f\colon A \to B$.

$$\begin{array}{ccc} !A & \xrightarrow{d_A} & !A\otimes !A \\ {\scriptstyle !f}\downarrow & & \downarrow{\scriptstyle !f\otimes !f} \\ !B & \xrightarrow[d_B]{} & !B\otimes !B \end{array} \qquad\qquad \begin{array}{ccc} I & \xrightarrow{\lambda^{-1}} & I\otimes I \\ {\scriptstyle \mathsf{m}_I}\downarrow & & \downarrow{\scriptstyle \mathsf{m}_I\otimes \mathsf{m}_I} \\ !I & \xrightarrow[d_I]{} & !I\otimes !I \end{array}$$

$$\begin{array}{ccccc} !A\otimes !B & \xrightarrow{d_A\otimes d_B} & (!A\otimes !A)\otimes(!B\otimes !B) & \xrightarrow{iso} & (!A\otimes !B)\otimes(!A\otimes !B) \\ {\scriptstyle \mathsf{m}_{A,B}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{m}_{A,B}\otimes \mathsf{m}_{A,B}} \\ !(A\otimes B) & & \xrightarrow[d_{A\otimes B}]{} & & !(A\otimes B)\otimes !(A\otimes B) \end{array}$$

Where *iso* represents a combination of the natural isomorphisms.[9] Requiring that $(!A, d_A, e_A)$ forms a commutative comonoid amounts to requiring that the following three diagrams commute.

[9]For example, we could take

$$iso \stackrel{\text{def}}{=} \alpha^{-1}_{!A,!A,(!B\otimes !B)}; \mathsf{id}\otimes\gamma_{!A,!B\otimes !B}; \mathsf{id}\otimes\alpha^{-1}_{!B,!B,!A}; \mathsf{id}\otimes(\mathsf{id}\otimes\gamma_{!B,!A}); \alpha_{!A,!B,!A\otimes !B}.$$

$$\frac{}{x{:}\,!A \rhd \mathsf{promote}\ x\ \mathsf{for}\ y\ \mathsf{in}\ \mathsf{derelict}(y) = x{:}\,!A}\ \textit{Comonad}$$

$$\frac{\begin{array}{l}\Gamma_1 \rhd M_1{:}\,!A_1 \\ \Gamma_n \rhd M_n{:}\,!A_n \qquad x_2{:}\,!A_2,\ldots,x_n{:}\,!A_n \rhd N{:}\,B\end{array}}{\begin{array}{rl}\Gamma_1,\ldots,\Gamma_n \rhd & \mathsf{promote}\ (M_1,\ldots,M_n)\ \mathsf{for}\ (x_1,\ldots,x_n)\ \mathsf{in}\ \mathsf{discard}\ x_1\ \mathsf{in}\ N \\ = & \mathsf{discard}\ M_1\ \mathsf{in} \\ & \mathsf{promote}\ (M_2,\ldots,M_n)\ \mathsf{for}\ (x_2,\ldots,x_n)\ \mathsf{in}\ N{:}\,!B\end{array}}\ \textit{Coalgebra}_1$$

$$\frac{\begin{array}{l}\Gamma_1 \rhd M_1{:}\,!A_1 \\ \Gamma_n \rhd M_n{:}\,!A_n \qquad y{:}\,!A_1, z{:}\,!A_1, x_2{:}\,!A_2,\ldots,x_n{:}\,!A_n \rhd N{:}\,B\end{array}}{\begin{array}{rl}\Gamma_1,\ldots,\Gamma_n \rhd & \mathsf{promote}\ (M_1,\ldots,M_n)\ \mathsf{for}\ (x_1,\ldots,x_n)\ \mathsf{in}\ \mathsf{copy}\ x_1\ \mathsf{as}\ y,z\ \mathsf{in}\ N \\ = & \mathsf{copy}\ M_1\ \mathsf{as}\ y',z'\ \mathsf{in} \\ & \mathsf{promote}\ (y',z',M_2,\ldots,M_n)\ \mathsf{for}\ (y,z,x_2,\ldots,x_n) \\ & \ \mathsf{in}\ N{:}\,!B\end{array}}\ \textit{Coalgebra}_2$$

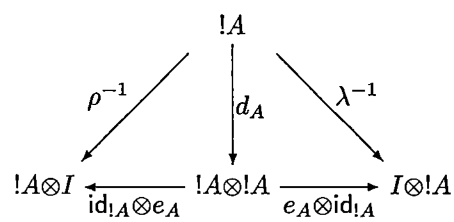
$$\frac{\Delta \rhd M{:}\,!A \qquad \Gamma, y{:}\,!A \rhd N{:}\,B}{\Gamma,\Delta \rhd \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ \mathsf{discard}\ x\ \mathsf{in}\ N = N[y := M]{:}\,B}\ \textit{Comonoid}_1$$

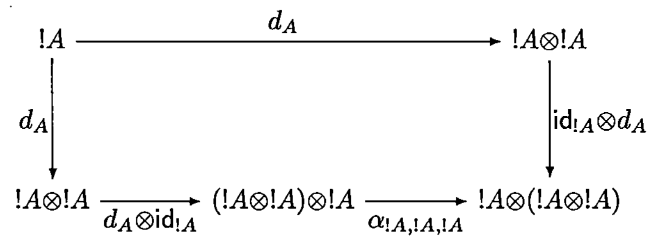
$$\frac{\Delta \rhd M{:}\,!A \qquad \Gamma, x{:}\,!A \rhd N{:}\,B}{\Gamma,\Delta \rhd \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ \mathsf{discard}\ x\ \mathsf{in}\ N = N[y := M]{:}\,B}\ \textit{Comonoid}_2$$

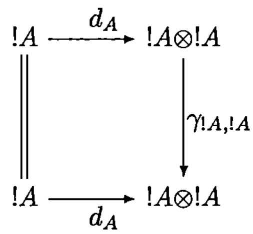
$$\frac{\Delta \rhd M{:}\,!A \qquad \Gamma, x{:}\,!A, y{:}\,!A \rhd N{:}\,B}{\Gamma,\Delta \rhd \mathsf{copy}\ M\ \mathsf{as}\ x,y\ \mathsf{in}\ N = \mathsf{copy}\ M\ \mathsf{as}\ y,x\ \mathsf{in}\ N{:}\,B}\ \textit{Comonoid}_3$$

$$\frac{\Delta \rhd M{:}\,!A \qquad \Gamma, x{:}\,!A, y{:}\,!A, z{:}\,!A \rhd N{:}\,B}{\begin{array}{rl}\Gamma,\Delta \rhd & \mathsf{copy}\ M\ \mathsf{as}\ x,w\ \mathsf{in}\ \mathsf{copy}\ w\ \mathsf{as}\ y,z\ \mathsf{in}\ N \\ & = \mathsf{copy}\ M\ \mathsf{as}\ w,z\ \mathsf{in}\ \mathsf{copy}\ w\ \mathsf{as}\ x,y\ \mathsf{in}\ N{:}\,B\end{array}}\ \textit{Comonoid}_4$$

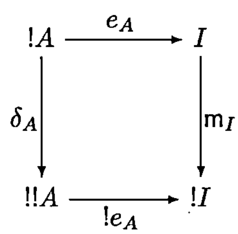
Figure 4.5: 'Categorical' Equations in Context



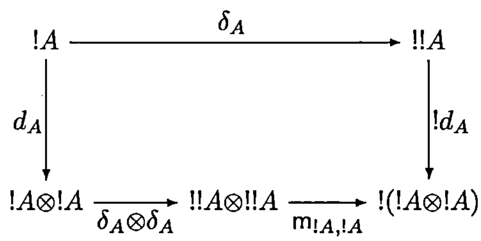




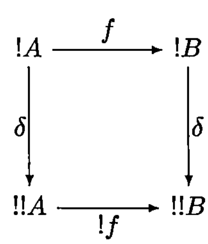
We have that $(!A, \delta)$ and (I, m_I) are coalgebras (Propositions 7 and 8). Requiring that e_A is a coalgebra morphism amounts to requiring that the following diagram commutes.



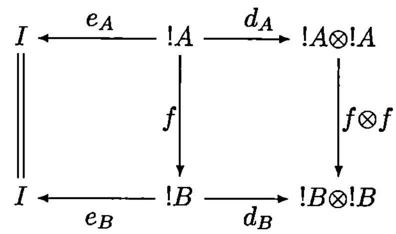
We have that also $(!A\otimes!A, (\delta\otimes\delta; \mathsf{m}_{A,A}))$ is a coalgebra (Proposition 9). Requiring that d_A is a coalgebra morphism amounts to requiring that the following diagram commutes.



We also require that all coalgebra morphisms between *free* coalgebras are also comonoid morphisms. Thus given a coalgebra morphism f, between the free coalgebras $(!A, \delta)$ and $(!B, \delta)$, i.e. which makes the following diagram commute.



Then it is also a comonoid morphism between the comonoids $(!A, e_A, d_A)$ and $(!B, e_B, d_B)$, i.e. it makes the following diagram commute.



These amount to some strong conditions on the model and we shall explore some of their consequences. Firstly notice that in the definition a comonoid structure is only required for the *free* coalgebra structure. We shall consider the question of a comonoid structure on *any* coalgebra.

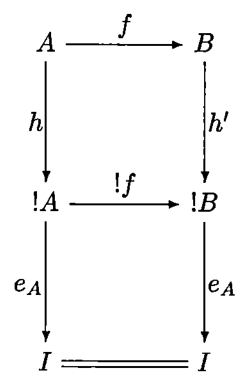
Definition 36. In the category of coalgebras for a Linear category, for *any* coalgebra $(A, h_A: A \to !A)$ we define two morphisms, $d^!: A \to A\otimes A$ and $e^!: A \to I$ in terms of the free coalgebra and comonoid structures.

$$
\begin{aligned}
d^! &\stackrel{\text{def}}{=} h_A; d_A; \varepsilon_A\otimes\varepsilon_A \\
e^! &\stackrel{\text{def}}{=} h_A; e_A
\end{aligned}
$$

We shall consider some properties of these morphisms.

Lemma 15. In the category of coalgebras for a Linear category $d^!$ and $e^!$ are natural transformations.

Proof. Take a morphism $f: A \to B$. For $e^!$ to be a natural transformation, the following diagram must commute.



The upper square commutes by f being a coalgebra morphism (as it is a morphism in the category of coalgebras) and the lower square commutes by the naturality of e_A. For $d^!$ to be a natural transformation, the following diagram must commute.

$$\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h'} \\
!A & \xrightarrow{\quad !f \quad} & !B \\
\Big\downarrow{\scriptstyle d_A} & & \Big\downarrow{\scriptstyle d_B} \\
!A\otimes!A & \xrightarrow{!f\otimes!f} & !B\otimes!B \\
\Big\downarrow{\scriptstyle \varepsilon_A\otimes\varepsilon_A} & & \Big\downarrow{\scriptstyle \varepsilon_A\otimes\varepsilon_A} \\
A\otimes A & \xrightarrow{f\otimes f} & B\otimes B
\end{array}$$

The upper square commutes by f being a coalgebra morphism; the middle square commutes by the naturality of d and the lower square by naturality of ε. ■

Corollary 3. Whenever $f:(A,h) \to (A',h')$ is a coalgebra morphism between *any* two coalgebras then it is a comonoid morphism.

We shall define a notion which is opposite to that given by MacLane [53].

Definition 37. Given two parallel morphisms $f,g: B \to C$ in a category $\mathbb{C}$, a *cofork* is a morphism $c: A \to B$ such that $c; f = c; g$, i.e. the two paths in the following diagram are equal.

$$A \xrightarrow{\quad c \quad} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

We can see that it is easy to extend this notion to an *equalizer* of two morphisms.[10] We can also isolate a special case of a cofork.

Definition 38. An *identity cofork* of two morphisms $f,g: B \to A$ is a morphism $c: A \to B$ such that $c; f = c; g = \mathsf{id}_A$.[11]

Using these notions, for example, we can reformulate the definition of a comonad.

Definition 39. A comonad on a category $\mathbb{C}$ is a triple $(!,\varepsilon,\delta)$ where $!$ is an endofunctor and $\delta: ! \dot{\to} !^2$ and $\varepsilon: ! \dot{\to} Id$ are natural transformations such that δ_A is a cofork of $\delta_{!A}$ and $!\delta_A$ and is also the identity cofork of ε_A and $!\varepsilon_A$.

Let us use this notion to isolate some useful coforks within our notion of a Linear category.

Proposition 11. Given a Linear category, $\mathbb{C}$, the morphism $(h_A; d_A): A \to !A\otimes!A$ is a cofork of the two morphisms $(\varepsilon_A\otimes\varepsilon_A; \mathsf{id}_A\otimes h_A)$ and $(\varepsilon_A\otimes\mathsf{id}_{!A}): !A\otimes!A \to A\otimes!A$.

---

[10] A morphism $c: A \to B$ is an *equalizer* of two morphisms $f,g: B \to C$ if it is a cofork and for all morphisms $h: D \to B$, $h; f = h; g$ implies that there exists a unique morphism $k: D \to A$ such that $h = k; c$.

[11] Alternative terminology due to Linton [56] is that if an identity cofork exists for a pair of morphisms, then the pair is said to be *reflexive*.

Proof. We shall prove this equationally rather than use diagrams.

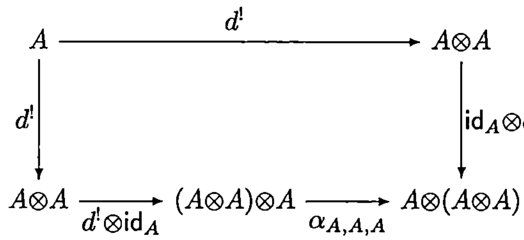
$$
\begin{aligned}
h_A; d_A; \varepsilon\otimes\varepsilon; \mathsf{id}\otimes h_A &= d_A; h\otimes h; \varepsilon\otimes\varepsilon; \mathsf{id}\otimes h_A && \text{Proposition 10 and Corollary 3}\\
&= d_A; (h;\varepsilon)\otimes(h;\varepsilon); \mathsf{id}\otimes h_A &&\\
&= d_A; \mathsf{id}\otimes h_A && \text{Definition of a coalgebra}\\
&= d_A; (h;\varepsilon)\otimes h_A &&\\
&= d_A; (h;\varepsilon)\otimes(h_A; \mathsf{id}_{!A}) &&\\
&= d_A; (h\otimes h); (\varepsilon\otimes\mathsf{id}_{!A}) &&\\
&= h_A; d_A; (\varepsilon\otimes\mathsf{id}_{!A}) && h \text{ is a comonoid morphism}
\end{aligned}
$$

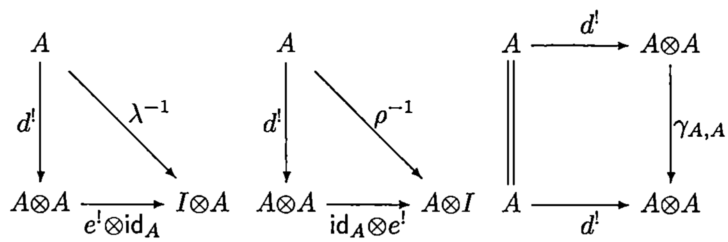
■

Thus we have a commutative comonoid structure on the free coalgebras by definition, and we have constructed in Definition 36 candidates for a commutative comonoid structure on any coalgebra. We shall now prove that these candidates do indeed form a commutative comonoid structure on any coalgebra.

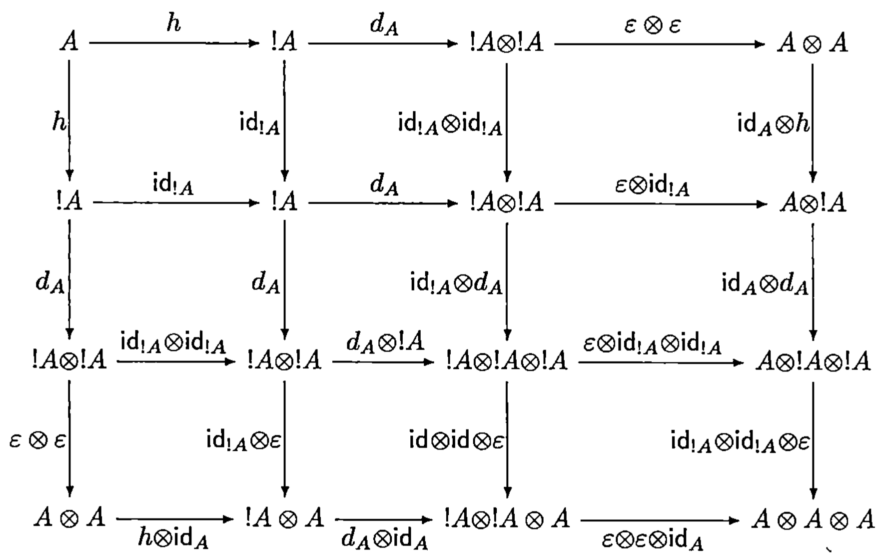
Proposition 12. In the category of coalgebras for a Linear category, the natural transformations $d^!$ and $e^!$ form a commutative comonoid.

Proof. For $(A, d^!, e^!)$ to be a commutative comonoid we need the four following diagrams to commute.



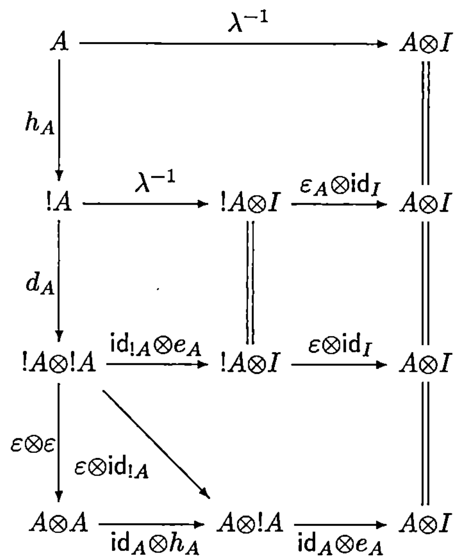


Using the definitions of the morphisms $d^!$ and $e^!$, we can show that these diagrams commute. We take the diagrams in turn; considering the first diagram we shall make the usual simplification and ignore the associativity morphism.



The upper left square commutes trivially. The upper middle square commutes because of the naturality of d. The upper right square does *not* commute, but rather follows from Proposition 11. The middle left square commutes from the naturality of d. The middle square commutes by definition of a comonoid. The middle right square commutes by the naturality of d. The lower left square does *not* commute but similarly follows from Proposition 11. The lower middle square commutes by the naturality of d and the lower right square commutes trivially.

Considering the second diagram

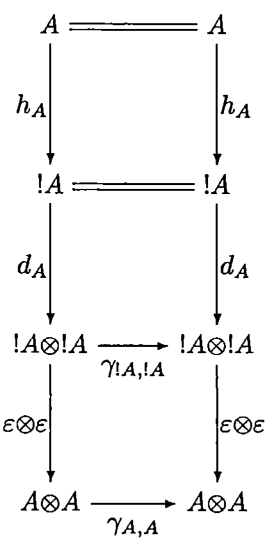


The upper square commutes using the following reasoning

$$\begin{aligned}
h_A;\lambda^{-1};\varepsilon\otimes\mathsf{id}_I &= \lambda^{-1};h_A\otimes\mathsf{id}_I;\varepsilon\otimes\mathsf{id}_I && \text{Naturality of } \lambda^{-1}\\
&= \lambda^{-1};(h_A;\varepsilon)\otimes(\mathsf{id}_I;\mathsf{id}_I)\\
&= \lambda^{-1};(\mathsf{id}_A)\otimes(\mathsf{id}_I) && \text{Definition of coalgebra}\\
&= \lambda^{-1}
\end{aligned}$$

The middle left square commutes by definition of a comonoid, and the middle right commutes trivially. The lower left triangle does *not* commute, but we have from Proposition 11 that $h_A;d_A$ is a cofork of $\varepsilon\otimes\varepsilon;\mathsf{id}_A\otimes h_A$ and $\varepsilon\otimes\mathsf{id}_{!A}$. The lower right square commutes trivially.

The symmetric case for ρ^{-1} instead of λ^{-1} works similarly. Considering the fourth diagram:

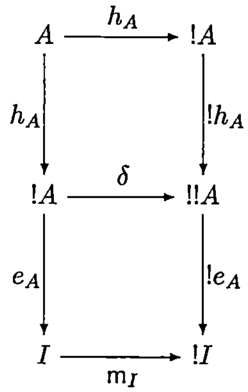


The upper square commutes trivially. The middle square commutes because $(!A, d_A, e_A)$ is a commutative comonoid by definition. The lower square commutes by naturality of γ. ∎

We have in the definition of a Linear category, that the comonoid morphisms, d_A and e_A, are also coalgebra morphisms. Let us consider whether our candidates, $e^!$ and $d^!$, have this property.

Proposition 13. The morphism $e^!$ is a coalgebra morphism between the coalgebras (A, h) and (I, m_I).

Proof. For $e^!$ to be a coalgebra morphism the following diagram must commute.



The upper square commutes by definition of a coalgebra. The lower square commutes since e_A is a coalgebra morphism (by definition). ∎

Proposition 14. The morphism $(h_A; d_A)$ is a cofork of the morphisms $(\varepsilon\otimes\varepsilon; h_A\otimes h_A)$ and $(\delta\otimes\delta; !\varepsilon\otimes!\varepsilon) \colon !A\otimes!A \to !A\otimes!A$.

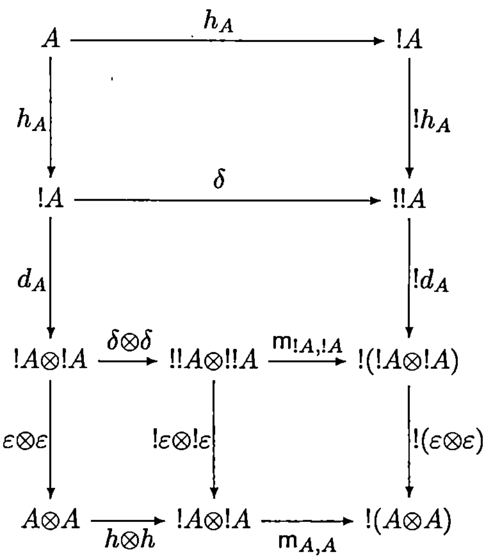
Proof. Again we shall prove this equationally rather than with the use of diagrams.

$$\begin{array}{rcll}
h_A; d_A; \varepsilon\otimes\varepsilon; h\otimes h & = & d; h\otimes h; \varepsilon\otimes\varepsilon; h\otimes h & \text{Proposition 10 and Corollary 3} \\
& = & d; (h;\varepsilon)\otimes(h;\varepsilon); h\otimes h & \\
& = & d; h\otimes h & \text{Definition of a coalgebra} \\
& = & h; d & \\
& = & h; d; \mathsf{id}\otimes\mathsf{id} & \\
& = & h; d; (\delta; !\varepsilon)\otimes(\delta; !\varepsilon) & \text{Definition of a comonad} \\
& = & h; d; (\delta\otimes\delta); (!\varepsilon\otimes!\varepsilon) &
\end{array}$$

∎

Proposition 15. The morphism $d^!$ is a coalgebra morphism between the coalgebras (A, h_A) and $(A\otimes A, (h_A\otimes h_A; \mathsf{m}_{A,A}))$.

Proof. For $d^!$ to be a coalgebra morphism the following diagram must commute.



The upper square commutes by the definition of a coalgebra. The middle square commutes as d_A is a coalgebra morphism between the coalgebras $(!A, \delta)$ and $(!A\otimes !A, (\delta\otimes\delta; \mathsf{m}_{!A,!A}))$. The lower left square does *not* commute, but follows from Proposition 14.[12] The lower right square commutes from the naturality of m. ■

Thus we have shown that for any coalgebra we have two morphisms, $d^!$ and $e^!$, and that these are natural transformations, form a commutative comonoid and are coalgebra morphisms.

We can now state a important property of our categorical model: *soundness*

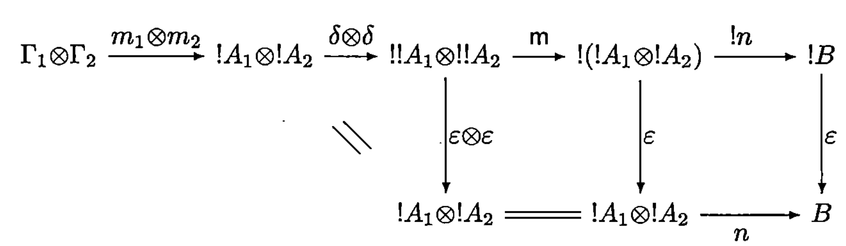
Theorem 24. A Linear category, $\mathbb{C}$, is a model of the LTC-theory $(\mathcal{L}, \mathcal{A})$.

Proof. To prove that $\mathbb{C}$ is a structure for the LTC-theory we simply proceed by induction over the structure of a given term in context. To verify that $\mathbb{C}$ models the equations in context in $\mathcal{A}$ we simply check them exhaustively. We shall simply give two examples.

1. Consider the following equation in context (which is slightly less general for clarity):

$$\frac{\Gamma_1 \rhd M_1 \colon !A_1 \qquad \Gamma_2 \rhd M_2 \colon !A_2 \qquad x_1 \colon !A_1, x_2 \colon !A_2 \rhd N \colon B}{\Gamma_1, \Gamma_2 \rhd \mathsf{derelict}(\mathsf{promote}\ M_1, M_2\ \mathsf{for}\ x_1, x_2\ \mathsf{in}\ N) = N[x_1 := M_1, x_2 := M_2] \colon B}\ Der_{Eq}$$

   This term in context amounts to the following diagram:



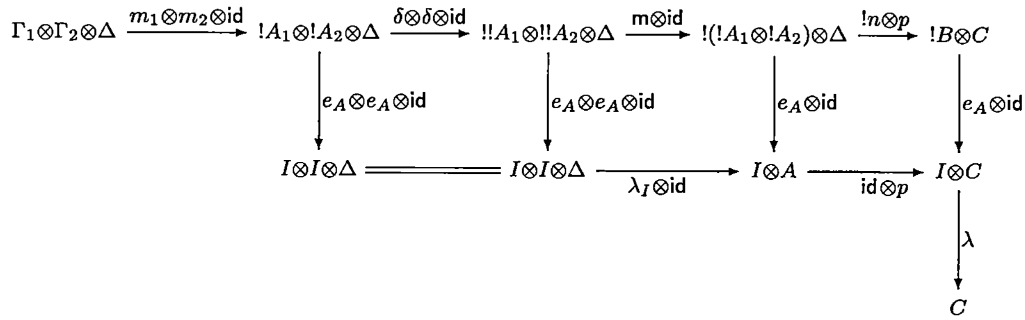
   The left triangle commutes by the definition of a comonad. The middle square commutes by the fact that ε is a monoidal natural transformation and the right hand square commutes by naturality of ε.

[12] In fact it would be disastrous were this square to commute. We would then have that $h_A; \varepsilon = \mathsf{id}_{!A} = \varepsilon; h_A$. In other words we would have succeeded in collapsing the model to the extent that $A \cong !A$.

2. Consider the following equation in context:

$$\frac{\Gamma_1 \rhd M_1 : !A_1 \quad \Gamma_2 \rhd M_2 : !A_2 \quad x_1 : !A_1, x_2 : !A_2 \rhd N : B \quad \Delta \rhd P : C}{\begin{array}{rl}\Gamma_1, \Gamma_2, \Delta \;\rhd & \mathsf{discard}\ (\mathsf{promote}\ M_1, M_2\ \mathsf{for}\ x_1, x_2\ \mathsf{in}\ N)\ \mathsf{in}\ P \\ & = \mathsf{discard}\ M_1, M_2\ \mathsf{in}\ P : C\end{array}}\ \mathit{Disc}_{Eq}$$

This term in context amounts to the following diagram:



The left hand square commutes as δ is a comonoid morphism by definition. The middle square commutes as e_A is a monoidal natural transformation. The right hand square commutes by naturality of e.

■

In fact, it is the case that a Linear category models *all* the reduction rules from Chapter 3, if they are regarded as equations in context. For example, consider the following equation in context which arises from the cut elimination process.

$$\frac{\Gamma_1 \rhd M_1 : !A \quad \Delta \rhd M_2 : !B \quad \Theta \rhd M_3 : !C \quad x : !A, y : !C \rhd N : D}{\begin{array}{rl}\Gamma, \Delta, \Theta \;\rhd & \mathsf{promote}\ M_1, (\mathsf{discard}\ M_2\ \mathsf{in}\ M_3)\ \mathsf{for}\ x, y\ \mathsf{in}\ N \\ & = \mathsf{discard}\ M_2\ \mathsf{in}\ (\mathsf{promote}\ M_1, M_3\ \mathsf{for}\ x, y\ \mathsf{in}\ N)\ : !D\end{array}}$$

Categorically this amounts to the following equational reasoning

$$\begin{array}{lll}
\multicolumn{3}{l}{m_1 \otimes m_2; \mathsf{id}\otimes\delta; \mathsf{id}\otimes !n; \delta\otimes\delta; \mathsf{m}_{!A,!A}; !p} \\
= & m_1 \otimes m_2; \mathsf{id}\otimes\delta; \delta\otimes\mathsf{id}; \mathsf{id}_{\otimes} !n; \mathsf{id}\otimes\delta; \mathsf{m}_{!A,!C}; !p & \\
= & m_1 \otimes m_2; \mathsf{id}\otimes\delta; \delta\otimes\mathsf{id}; \mathsf{id}\otimes\delta; \mathsf{id}_{\otimes} !!n; \mathsf{m}_{!A,!C}; !p & \text{Naturality of } \delta. \\
= & m_1 \otimes m_2; \mathsf{id}\otimes\delta; \delta\otimes\mathsf{id}; \mathsf{id}\otimes\delta; \mathsf{m}; !(\mathsf{id}\otimes !n); !p & \text{Naturality of m.} \\
= & m_1 \otimes m_2; \delta\otimes\delta; \mathsf{id}\otimes\delta; \mathsf{m}; !(\mathsf{id}\otimes !n); !p & \\
= & m_1 \otimes m_2; \delta\otimes\delta; \mathsf{id}\otimes !\delta; \mathsf{m}; !(\mathsf{id}\otimes !n); !p & \text{Def of a comonad.} \\
= & m_1 \otimes m_2; \delta\otimes\delta; \mathsf{m}; !(\mathsf{id}\otimes\delta); !(\mathsf{id}\otimes !n); !p &
\end{array}$$

The other reduction rules follow by similar (trivial) reasoning.

5 An Example Linear Category

There are many categories which satisfy the definition of a Linear category. In this section we shall detail just one, although from a computer science perspective, it is an extremely important one. We shall consider the category of domains and strict morphisms. This is an important example, as it is *not* a CCC, and hence, not directly a model of the extended λ-calculus. The fact that it is a Linear category, along with the Girard translation, shows us how it can be considered as a model for the extended λ-calculus.

5.1 The Category $\mathcal{D}om_s$

This is the category of pointed ω-cpos (domains) and strict morphisms. We take $A \otimes B$ to be the smash product of the domains A and B, $A \multimap B$ to be the set of strict morphisms from A to B ordered pointwise, I to be the two point domain $(\mathbf{2})$, $A \& B$ to be the cartesian product and $A \oplus B$ to be the coalesced sum. We shall use the lifting construction to represent $!A$. The lifting of a domain A, written $A_\bot$, is the set $\{(0, a) | a \in A\} \cup \{\bot\}$ as a carrier and is ordered by $x \sqsubseteq y$ if $x = \bot$ or $x = (0, a)$ and $y = (0, a')$ and $a \sqsubseteq_A a'$.

We can define two constructions: *delta*: $A_\bot \to A_{\bot\bot}$ which maps $\bot$ to $\bot$ and $(0, a)$ to $(0, (0, a))$ and *eps*: $A_\bot \to A$ which maps $\bot$ to $\bot$ and $(0, a)$ to a. It is not hard to see that the lifting functor, together with *eps* and *delta* form a comonad on $\mathcal{D}om_s$. We can then define a morphism $m_I: I \to I_\bot$ which maps $\bot$ to $\bot$ and 0 to $(0, 0)$. We can also define a family of morphisms $m_{A,B}: A_\bot \otimes B_\bot \to (A \otimes B)_\bot$ which map $\bot$ to $\bot$ and $((0, a), (0, b))$ to $(0, \bot)$ if either $a = \bot$ or $b = \bot$ and to $(0, (a, b))$ otherwise. We can also define two other families of morphisms: $d_A: A_\bot \to A_\bot \otimes A_\bot$ which maps $\bot$ to $\bot$ and $(0, a)$ to $((0, a), (0, a))$ and $e_A: A_\bot \to \mathbf{2}$ which maps $\bot$ to $\bot$ and $(0, a)$ to 0. It is then routine to check that these constructions satisfy the definition of a Linear category.

6 Comparison: Seely's Model

The most well-known alternative definition of a categorical model of **ILL** is that due to Seely [69]. Rather than use coalgebras, Seely's model provides a pair of natural isomorphisms which relates the tensor product and the categorical product. It should be noted that this means that products *must* exist to model the exponential. This is an important difference between Seely's model and a Linear category where the exponential can be fully given without reference to products (and indeed was done so in an earlier presentation [15]).

In this section we shall consider in detail Seely's proposal and, where appropriate, compare it with that given in Definition 35. First let us recall Seely's definition.

Definition 40. A *Seely category*, $\mathbb{C}$, consists of:

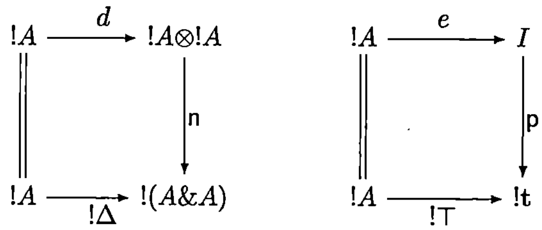
1. A SMCC with finite products, together with a comonad $(!, \varepsilon, \delta)$.
2. For each object A of $\mathbb{C}$, $(!A, d_A, e_A)$ is a comonoid with respect to the tensor product.
3. There exists natural isomorphisms $\mathsf{n}: !A \otimes !B \xrightarrow{\sim} !(A \& B)$ and $\mathsf{p}: I \xrightarrow{\sim} !\mathsf{t}$.
4. The functor ! takes the comonoid structure of the cartesian product to the comonoid structure of the tensor product.

The naturality of n amounts to the following diagram commuting for morphisms $f: A \to C$ and $g: B \to D$.

$$\begin{array}{ccc}
!A \otimes !B & \xrightarrow{\mathsf{n}} & !(A \& B) \\
{\scriptstyle !f \otimes !g}\downarrow & & \downarrow{\scriptstyle !(f \& g)} \\
!C \otimes !D & \xrightarrow[\mathsf{n}]{} & !(C \& D)
\end{array}$$

Condition 4 (which seems to have been overlooked[13] by Barr [10] and Troelstra [75]) amounts to requiring that the following two diagrams commute.

[13] Asperti and Longo [6, Lemma 5.5.4] (falsely) claim it holds automatically.



We have appropriate candidates for the natural isomorphisms n, n^{-1}, p and p^{-1} in the interpretation of the following proofs.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{!A \vdash A}\,\textit{Dereliction}}{!A, !B \vdash A}\,\textit{Weakening} \qquad \dfrac{\dfrac{B \vdash B}{!B \vdash B}\,\textit{Dereliction}}{!A, !B \vdash B}\,\textit{Weakening}}{!A, !B \vdash A\&B}\,(\&_{\mathcal{R}})}{!A, !B \vdash !(A\&B)}\,\textit{Promotion}}{!A\otimes !B \vdash !(A\&B)}\,(\otimes_{\mathcal{L}})$$

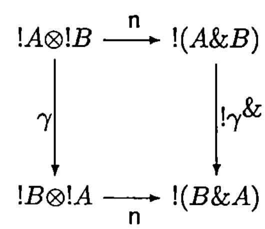
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A\&B \vdash A}\,(\&_{\mathcal{L}})}{!(A\&B) \vdash A}\,\textit{Dereliction}}{!(A\&B) \vdash !A}\,\textit{Prom.} \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{A\&B \vdash B}\,(\&_{\mathcal{L}})}{!(A\&B) \vdash B}\,\textit{Dereliction}}{!(A\&B) \vdash !B}\,\textit{Prom.}}{!(A\&B), !(A\&B) \vdash !A\otimes !B}\,(\otimes_{\mathcal{R}})}{!(A\&B) \vdash !A\otimes !B}\,\textit{Cont.}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{t}}\,(\mathsf{t}_{\mathcal{R}})}{\vdash !\mathsf{t}}\,\textit{Promotion}}{I \vdash !\mathsf{t}}\,(I_{\mathcal{L}}) \qquad\qquad \dfrac{\dfrac{}{\vdash I}\,(I_{\mathcal{R}})}{!\mathsf{t} \vdash I}\,\textit{Weakening}$$

Essentially the main difference arises in the interpretation of the *Promotion* rule. With a Seely category this is interpreted:

$$\begin{array}{l}[\![\Gamma_1, \ldots, \Gamma_n \rhd \mathsf{promote}\ M_1, \ldots, M_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ N : !B]\!] \\ \stackrel{\text{def}}{=} \quad [\![\Gamma_1 \rhd M_1 : !A_1]\!] \otimes \ldots \otimes [\![\Gamma_n \rhd M_n : !A_n]\!]; \mathsf{n}; \delta; !\mathsf{n}^{-1}; !([\![x_1 : !A_1, \ldots, x_n : !A_n \rhd N : B]\!])\end{array}$$

Let us consider the n morphism. Since it maps the tensor product to the categorical product, we might expect it to preserve the symmetric monoidal structure of the tensor product (as a categorical product can be thought of as a tensor product). For example, we might expect it to preserve the symmetry of the tensor product.



(4.39)

(Where we write $\gamma^{\&}$ to denote the symmetry morphism for the categorical product.) However given Seely's definition this diagram *does not necessarily commute.* Condition 4 of the definition only entails that the following diagram commutes.

$$\begin{array}{ccc}
!A\otimes!A & \xrightarrow{\mathsf{n}} & !(A\&A) \\
\Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle !\gamma^{\&}} \\
!A\otimes!A & \xrightarrow[\mathsf{n}]{} & !(A\&A)
\end{array}$$

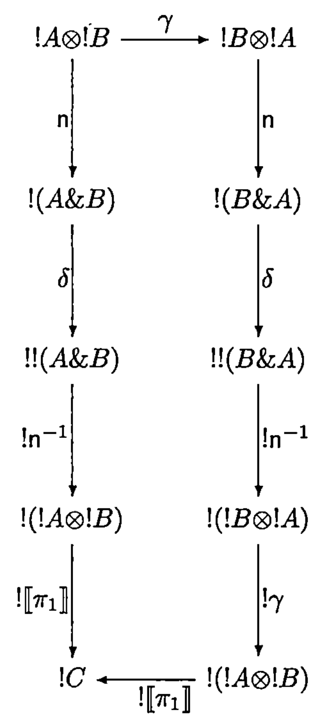
Unfortunately, we can find situations where we require diagram 4.39. Consider the following two proofs (given in the sequent calculus formulation).

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ !A, !B \vdash C\end{array}}{!B, !A \vdash C}\,\textit{Exchange}}{!B, !A \vdash\, !C}\,\textit{Promotion}}{!A, !B \vdash\, !C}\,\textit{Exchange} \qquad\qquad \dfrac{\begin{array}{c}\pi_1\\ !A, !B \vdash C\end{array}}{!A, !B \vdash\, !C}\,\textit{Promotion}$$

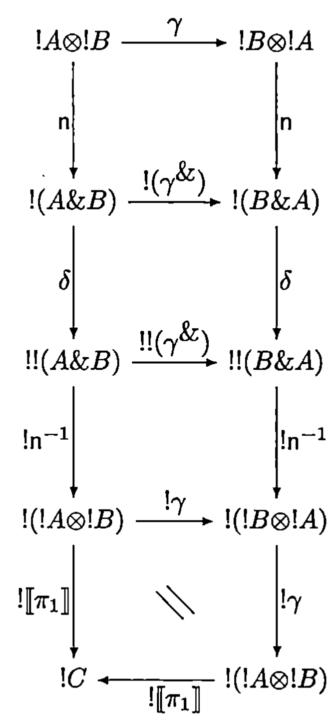
Clearly these two proofs are equivalent (they are given the same terms by the term assignment system from Chapter 3). Let us consider their interpretation. In a Linear category, their interpretations are equal as the following diagram commutes.

$$\begin{array}{ccc}
!A\otimes!B & \xrightarrow{\gamma} & !B\otimes!A \\
\Big\downarrow{\scriptstyle\delta\otimes\delta} & & \Big\downarrow{\scriptstyle\delta\otimes\delta} \\
!!A\otimes!!B & \xrightarrow{\gamma} & !!B\otimes!!A \\
\Big\downarrow{\scriptstyle\mathsf{m}} & & \Big\downarrow{\scriptstyle\mathsf{m}} \\
!(!A\otimes!B) & \xrightarrow{!\gamma} & !(!B\otimes!A) \\
\Big\downarrow{\scriptstyle ![\![\pi_1]\!]} & \diagdown\!\!\diagdown & \Big\downarrow{\scriptstyle !\gamma} \\
!C & \xleftarrow[{![\![\pi_1]\!]}]{} & !(!A\otimes!B)
\end{array}$$

The upper square commutes by naturality of γ, and the middle square commutes by definition of m. The left triangle commutes trivially and the right triangle commutes by definition of a SMC. In a Seely category the equivalence of the two proofs implies the following diagram commuting.



There seems to be no way of filling in this diagram using Seely's original definition. However if we include condition 4.39 given earlier, we can complete the diagram in the following way.



The upper square square commutes given our new condition, the middle square commutes by naturality of δ and the lower square again by our new condition. The two triangles commute as for the Linear category case.

Thus we might extend the definition of a Seely category to incorporate the fact that the n (and p) morphisms should preserve the symmetric monoidal structure of the tensor product. This amounts to four commuting diagrams which are given in Figure 4.6.

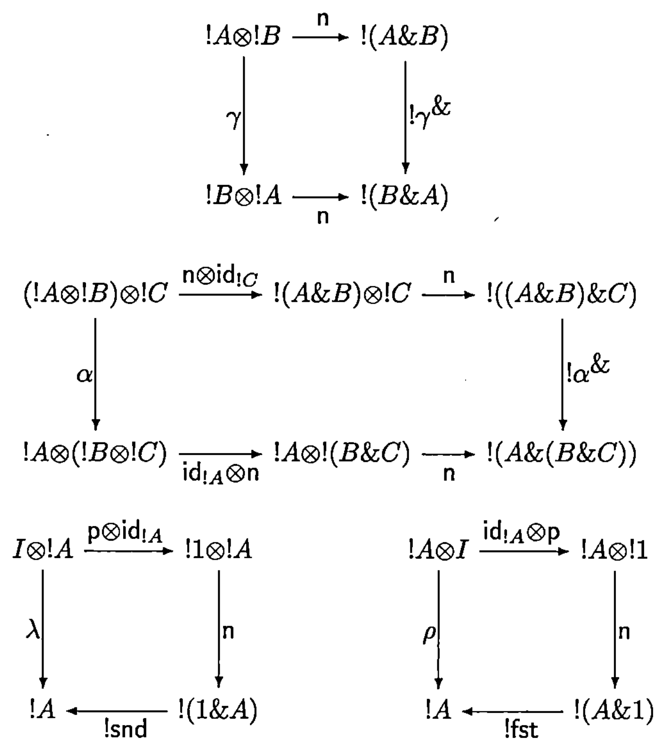


Figure 4.6: Four Additional Conditions for a Seely Category.

Given these new conditions let us now consider the property of soundness for a Seely category. Seely showed that all terms in context were modelled by a Seely category.

Proposition 16. (Seely) A Seely category, $\mathbb{C}$, is a *structure* for the LTC-theory $(\mathcal{L}, \mathcal{A})$.

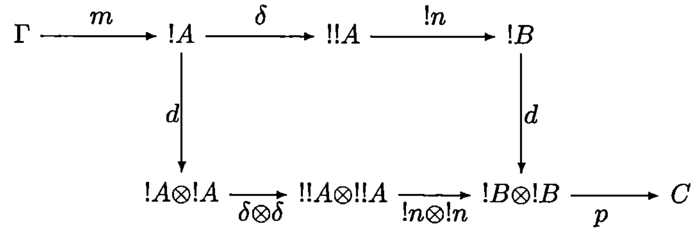
However for a Seely category to also be a *model* for a LTC-theory we need to show that the equations in context in $\mathcal{A}$ are modelled by equal morphisms. Unfortunately, for a Seely category we find that it is not true.

Lemma 16. Given a Seely category, $\mathbb{C}$, and the LTC-theory $(\mathcal{L}, \mathcal{A})$, it is *not* the case that for all equations in context, $\Gamma \rhd M = N : A$, in $\mathcal{A}$ that $[\![\Gamma \rhd M : A]\!] =_{\mathbb{C}} [\![\Gamma \rhd N : A]\!]$.

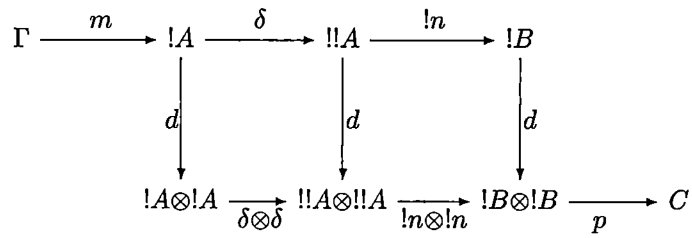
A counter-example is the $Copy_{Eq}$ equation in context. First let us recall the equation in context (in fact we only need use a simplified version where the promoted term has only one free variable).

$$\frac{\Gamma \rhd M : !A \qquad x : !A \rhd N : B \qquad y : !B, z : !B \rhd P : C}{\begin{array}{ll}\Gamma \rhd & \mathsf{copy}\ (\mathsf{promote}\ M\ \mathsf{for}\ x\ \mathsf{in}\ N)\ \mathsf{as}\ y, z\ \mathsf{in}\ P \\ & =\ \mathsf{copy}\ M\ \mathsf{as}\ x', x''\ \mathsf{in} \\ & \quad P[y := \mathsf{promote}\ x'\ \mathsf{for}\ x\ \mathsf{in}\ N, z := \mathsf{promote}\ x''\ \mathsf{for}\ x\ \mathsf{in}\ N] : C\end{array}}\ Copy_{Eq}$$

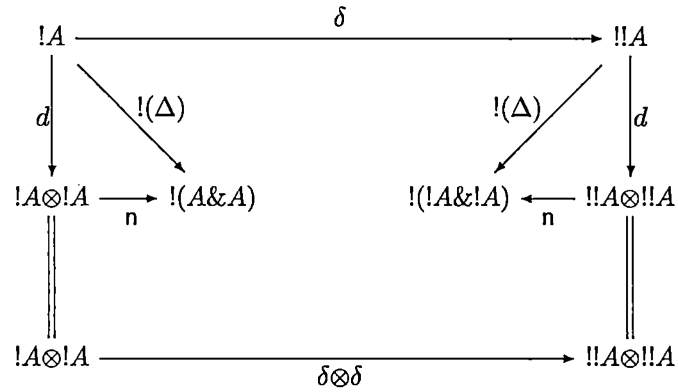
This equation in context implies the *same commuting diagram* for a Linear category as for a Seely category.



For a Linear category we can complete the diagram in the following way.



The left hand square commutes by the condition that all free coalgebra morphisms are comonoid morphisms. The right hand square commutes by naturality of d. Unfortunately it is not clear how to make this diagram commute for a Seely category. The right hand square commutes by naturality (as for a linear category), but then we can only reduce the left square to the following.



In fact we find that neither the $Copy_{Eq}$ nor the $Disc_{Eq}$ equations in context are modelled correctly. It is clear that we need to improve on Seely's original definition to obtain a sound model of **ILL**. Following a suggestion of Martin Hyland, we shall reconsider Seely's model in a slightly more abstract way. Let us consider some of the motivation behind the Seely construction.

First we shall recall a construction, the opposite of which (i.e. that generated by a monad) is known as the "Kleisli category" [53, Page 143].

Definition 41. Given a comonad, $(!, \varepsilon, \delta)$ on a category $\mathbb{C}$, we take all the objects A in $\mathbb{C}$ and for each morphism $f\colon !A \to B$ in $\mathbb{C}$ we take a new morphism $\hat{f}\colon A \to B$. The objects and morphisms form the *co-Kleisli category*, $\mathbb{C}_!$, where the composition of the morphisms $\hat{f}\colon A \to B$ and $\hat{g}\colon B \to C$ is defined by the following:

$$\hat{f};\hat{g} \stackrel{\text{def}}{=} (\widehat{\delta_A;!f};g)$$

It is easy to see that this has strong similarities with the Girard translation given in Chapter 2 of **IL** into **ILL** where the intuitionistic implication is decomposed: $(A \to B)^\circ \stackrel{\text{def}}{=} !(A^\circ) \multimap B^\circ$. In fact, as first shown by Seely [69], the co-Kleisli construction can be thought of as a categorical equivalent of the Girard translation.

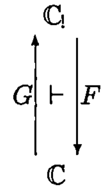
Proposition 17. (Seely) Given a Seely category, $\mathbb{C}$, the co-Kleisli category $\mathbb{C}_!$ is cartesian closed.

Proof. (Sketch) Given two objects A and B their exponent is defined to be $!A \multimap B$. Then we have the following sequence of isomorphisms.

$$\begin{array}{rcll}
\mathbb{C}_!(A\&B, C) & \cong & \mathbb{C}(!(A\&B), C) & \text{By definition} \\
& \cong & \mathbb{C}(!A \otimes !B, C) & \text{By use of the n isomorphism} \\
& \cong & \mathbb{C}(!A, !B \multimap C) & \text{By } \mathbb{C} \text{ having a closed structure} \\
& \cong & \mathbb{C}_!(A, !B \multimap C) & \text{By definition}
\end{array}$$

■

We know from Kleisli's construction that we have the following adjunction.



Seely's model arises from the desire to make the co-Kleisli category cartesian closed, which is achieved by including the n and p natural isomorphisms. This means we have an adjunction between a SMCC ($\mathbb{C}$) and a CCC ($\mathbb{C}_!$). As we can trivially view a CCC as a SMCC, we then have an adjunction between two SMCCs. We might expect that this is a *monoidal adjunction*.

Definition 42. A *monoidal adjunction*, $\langle F, G, \eta, \epsilon\rangle : \mathbb{C} \rightharpoonup \mathbb{D}$ is an adjunction where F and G are monoidal functors and η and ϵ are monoidal natural transformations.

Let us now state a new definition for a Seely-style category and then investigate some of its properties.

Definition 43. A *new-Seely category*, $\mathbb{C}$, consists of

1. a SMCC, $\mathbb{C}$, with finite products, together with
2. a comonad, $(!, \varepsilon, \delta)$, and
3. two natural isomorphisms, $\mathsf{n}: !A \otimes !B \xrightarrow{\sim} !(A \& B)$ and $\mathsf{p}: I \xrightarrow{\sim} !\mathsf{t}$

such that the adjunction, $\langle F, G, \eta, \varepsilon\rangle$, between $\mathbb{C}$ and $\mathbb{C}_!$ is a *monoidal* adjunction.

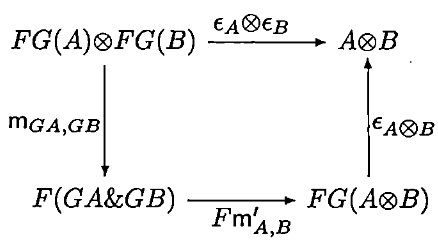
Assuming that F is monoidal gives us the following morphism and natural transformation:

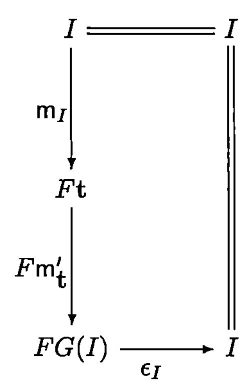
$$\mathsf{m}_I : I \to F\mathsf{t}$$
$$\mathsf{m}_{A,B} : FA \otimes FB \to F(A \& B)$$

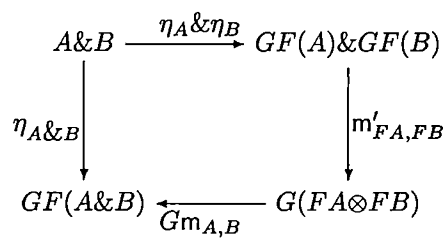
Assuming that G is monoidal gives us the following morphism and natural transformation;

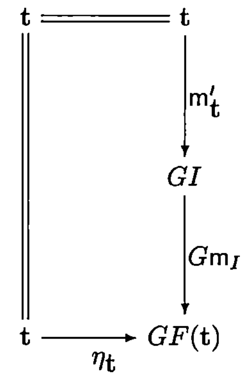
$$\mathsf{m}'_{\mathsf{t}} : \mathsf{t} \to GI$$
$$\mathsf{m}'_{A,B} : GA \& GB \to G(A \otimes B)$$

We also assume that ϵ and η are monoidal natural transformations. This amounts to the following diagrams









It is easy to see that m_I is Seely's morphism p and $\mathsf{m}_{A,B}$ is Seely's natural transformation n. In fact, we can define their inverses:

$$\mathsf{m}_I^{-1} \stackrel{\text{def}}{=} F\mathsf{m}'_{\mathsf{t}}; \epsilon_I : F\mathsf{t} \to I$$
$$\mathsf{m}_{A,B}^{-1} \stackrel{\text{def}}{=} F(\eta_A\&\eta_B); F\mathsf{m}'_{FA,FB}; \epsilon_{FA\otimes FB} : F(A\&B) \to FA\otimes FB$$

Lemma 17. (m_I^{-1} and m_I) and ($\mathsf{m}_{A,B}^{-1}$ and $\mathsf{m}_{A,B}$) are inverses.

Proof. By easy construction. ∎

Hence the monoidal adjunction itself provides the isomorphisms $!A\otimes !B \cong !(A\&B)$ and $I \cong !\mathsf{t}$. It is amusing to notice the following fact.

Proposition 18. Given a new-Seely category, $\mathbb{C}$, the diagrams in Figure 4.6 hold by the fact that F is a monoidal functor.

Proof. By writing our what it means for F to be a monoidal functor yields the relevant diagrams. ∎

As the co-Kleisli category is a CCC it has a canonical commutative comonoid structure, $(A, \triangle, \top)$, on all objects A. We can use this and the natural transformations arising from the monoidal adjunction to define a comonoid structure, $(F(A), d, e)$, on the objects of $\mathbb{C}$:

$$d \stackrel{\text{def}}{=} F(\triangle); \mathsf{m}_{A,A}^{-1} : F(A) \to F(A)\otimes F(A)$$
$$e \stackrel{\text{def}}{=} F(\top); \mathsf{m}_I^{-1} : F(A) \to I$$

Again taking $\mathsf{m}_{A,A}$ as $\mathsf{n}_{A,A}$ and m_I as p, these definitions amount to condition 4 of Seely's original definition. Additionally we know from basic category theory [53, Page 143–144] that the adjunction induces a (particular) comonad on $\mathbb{C}$.

Proposition 19. Given a new-Seely category, there is a comonad $(FG, F\eta_G, \epsilon)$ induced on C, which is precisely $(!, \varepsilon, \delta)$.

Thus we now know that we have at least as much structure as Seely's original definition. Of course, we have the extra structure of a monoidal adjunction and we shall see how these are precisely enough to produce a sound model for the LTC-theory. First of all, let us consider the comonad.

Proposition 20. Given a new-Seely category, the induced comonad $(FG, F\eta_G, \epsilon)$ is a *monoidal comonad* $(FG, F\eta_G, \epsilon, m_{A,B}, m_I)$.

Proof. Take the following:

$$m_I \stackrel{\text{def}}{=} \mathsf{m}_I; F\mathsf{m}'_{\mathsf{t}} : I \to FG(I)$$
$$m_{A,B} \stackrel{\text{def}}{=} \mathsf{m}_{GA,GB}; F\mathsf{m}'_{A,B} : FG(A)\otimes FG(A) \to FG(A\otimes B)$$

First we shall check the naturality of $m_{A,B}$: given two morphisms $f: A \to C$ and $g: B \to D$

$$
\begin{array}{ccccc}
FG(A)\otimes FG(B) & \xrightarrow{\mathsf{m}_{G(A),G(B)}} & F(G(A)\&G(B)) & \xrightarrow{F\mathsf{m}'_{A,B}} & FG(A\otimes B) \\
\Big\downarrow{\scriptstyle FG(f)\otimes FG(g)} & & \Big\downarrow{\scriptstyle F(Gf\&Gg)} & & \Big\downarrow{\scriptstyle FG(f\otimes g)} \\
FG(C)\otimes FG(D) & \xrightarrow[\mathsf{m}_{G(C),G(D)}]{} & F(G(C)\&G(D)) & \xrightarrow[F\mathsf{m}'_{C,D}]{} & FG(C\&D)
\end{array}
$$

The left hand square commutes by naturality of m and the right hand square commutes by naturality of m'.

Now we shall check that FG is a monoidal functor. As it is the composition of two monoidal functors, this is immediately obvious, and so a formal proof is omitted. We know that ϵ is a monoidal natural transformation by assumption. We also have that η is a monoidal natural transformation and it is a simple exercise to see that $F\eta_G$ is a monoidal functor.

■

Lemma 18. Given a new-Seely category, the comonoid morphisms $e: FG(A) \to I$ and $d: FG(A) \to FG(A)\otimes FG(A)$ are monoidal natural transformations.

Proof. We shall just give the proofs for d and e being natural transformations, the diagrams to prove that they are monoidal are laborious and omitted. For d to be a natural transformation, the following diagram must commute.

$$
\begin{array}{ccc}
FG(A) & \xrightarrow{FG(f)} & FG(B) \\
\Big\downarrow{\scriptstyle F(\triangle)} & & \Big\downarrow{\scriptstyle F(\triangle)} \\
F(GA\&GA) & \xrightarrow{F(Gf\&Gf)} & F(GB\&GB) \\
\Big\downarrow{\scriptstyle \mathsf{m}^{-1}} & & \Big\downarrow{\scriptstyle \mathsf{m}^{-1}} \\
FG(A)\otimes FG(B) & \xrightarrow[FG(f)\otimes FG(f)]{} & FG(B)\otimes FG(B)
\end{array}
$$

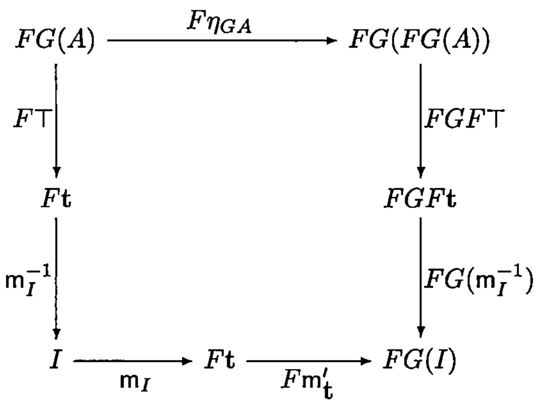
The upper square commutes by naturality of $\triangle$ and the lower by naturality of m^{-1}. For e to be a natural transformation, the following diagram must commute.

$$
\begin{array}{ccc}
FG(A) & \xrightarrow{FG(f)} & FG(B) \\
\Big\downarrow{\scriptstyle F(\top)} & & \Big\downarrow{\scriptstyle F(\top)} \\
F\mathsf{t} & = & F\mathsf{t} \\
\Big\downarrow{\scriptstyle \mathsf{m}_I^{-1}} & & \Big\downarrow{\scriptstyle \mathsf{m}_I^{-1}} \\
I & = & I
\end{array}
$$

■

Lemma 19. Given a new-Seely category, the comonoid morphisms $e: FG(A) \to I$ and $d: FG(A) \to FG(A) \otimes FG(A)$ are coalgebra morphisms.

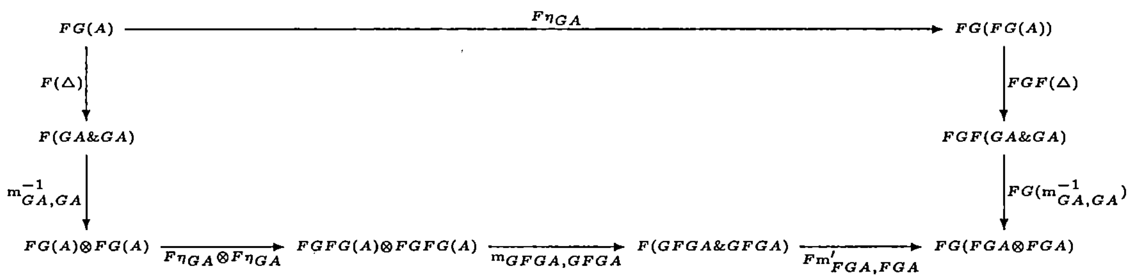
Proof. Requiring that $e: FG(A) \to I$ is a coalgebra morphism amounts to the following diagram commuting.



We prove that this commutes equationally

$$
\begin{aligned}
& F\eta_{GA}; FGF(\top); FG(\mathsf{m}_I^{-1}) \\
&= F\eta_{GA}; FGF(\top); FGF(\mathsf{m}'_{\mathsf{t}}); FG(\epsilon_I) \\
&= F\top; F\eta; FGF(\mathsf{m}'_{\mathsf{t}}); FG(\epsilon_I) && \text{Naturality} \\
&= F\top; F\mathsf{m}'_{\mathsf{t}}; F\eta_{GI}; FG(\epsilon_I) && \text{Naturality} \\
&= F\top; F\mathsf{m}'_{\mathsf{t}} && \text{Def of adjunction} \\
&= F\top; \mathsf{m}_I^{-1}; \mathsf{m}_I; F\mathsf{m}'_{\mathsf{t}}
\end{aligned}
$$

Requiring that $d: FG(A) \to FG(A) \otimes FG(A)$ is a coalgebra morphism amounts to the following diagram commuting.



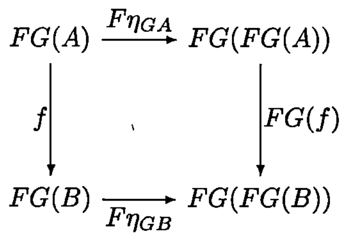
We prove that this commutes equationally.

$$
\begin{aligned}
& F(\triangle); \mathsf{m}^{-1}_{GA,GA}; F\eta_G \otimes F\eta_G; \mathsf{m}_{GFGA,GFGA}; F(\mathsf{m}'_{FGA,FGA}) \\
&= F(\triangle); \mathsf{m}^{-1}_{GA,GA}; F\eta_G \otimes F\eta_G; \mathsf{m}_{GFGA,GFGA}; F(\mathsf{m}'_{FGA,FGA}); F\eta_G; \epsilon \\
&= F(\triangle); F(\eta_G \& \eta_G); \mathsf{m}^{-1}; \mathsf{m}; F\mathsf{m}'; F\eta_G; \epsilon && \text{Nat of } \mathsf{m}^{-1} \\
&= F(\triangle); F(\eta_G \& \eta_G); F\mathsf{m}'; F\eta_G; \epsilon \\
&= F(\triangle); F(\eta_G \& \eta_G); F\eta; FGF(\mathsf{m}'); \epsilon && \text{Nat of } \eta \\
&= F(\triangle); F\eta; FGF(\eta_G \& \eta_G); FGF(\mathsf{m}'); \epsilon && \text{Nat of } \eta \\
&= F\eta; FGF(\triangle); FGF(\eta_G \& \eta_G); FGF(\mathsf{m}'); \epsilon && \text{Nat of } \eta \\
&= F\eta; FGF(\triangle); FG(\mathsf{m}^{-1})
\end{aligned}
$$

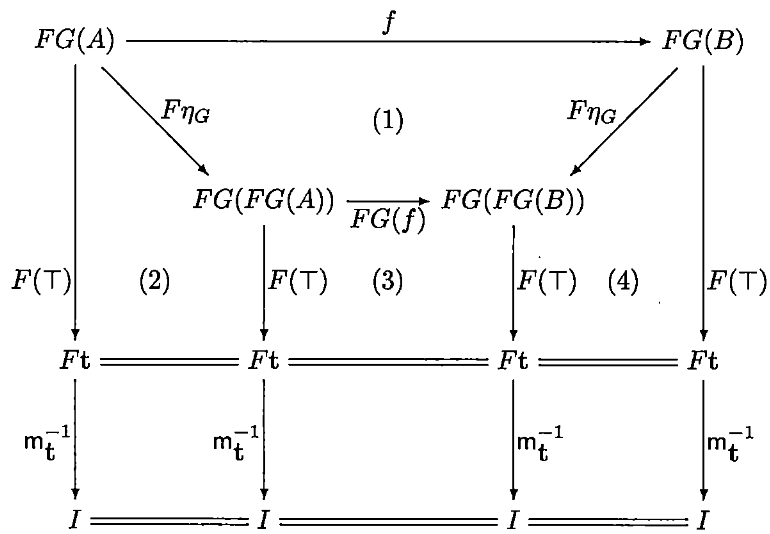
■

Lemma 20. Given a new-Seely category, if $f: (FG(A), F\eta_{GA}) \to (FG(B), F\eta_{GB})$ is a coalgebra morphism then it is also a comonoid morphism.

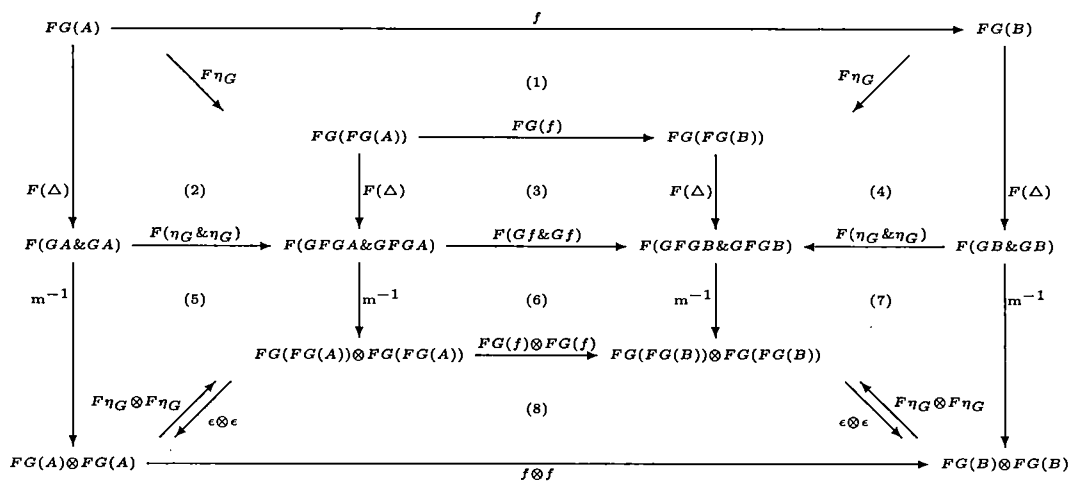
Proof. Assuming that f is a coalgebra morphism amounts to the following diagram commuting.



If f is a comonoid morphism then two diagrams must commute. Firstly:



Square (1) commutes by assumption. Squares (2), (3) and (4) commute by definition of $\top$. The lower squares all commute trivially. The second diagram is as follows.



Square (1) commutes by assumption. Squares (2), (3) and (4) commute by naturality of Δ. Squares (5), (6) and (7) commute by naturality of m^{-1} and, finally, square (8) commutes by naturality of ϵ. ■

Thus we can state the following relationship.

Theorem 25. Every new-Seely category is a Linear category.

Proof. From Proposition 20 and lemmas 18, 19 and 20. ■

We can hence show that a new-Seely category is a sound model for the LTC-theory.

Theorem 26. A new-Seely category, $\mathbb{C}$, is a model for the LTC-theory, $(\mathcal{L}, \mathcal{A})$.

Somewhat surprisingly, we find that the so-called Seely isomorphisms (n and p) exist in a Linear category with products.

Lemma 21. Given a Linear category with finite products we can define the following natural isomorphisms:

$$\begin{aligned} n &\stackrel{\text{def}}{=} \delta\otimes\delta; \mathsf{m}_{!A,!B}; !(\triangle); !((\mathsf{id}\otimes e_B)\&(e_A\otimes\mathsf{id})); !(\rho\&\lambda); !(\varepsilon\&\varepsilon)\colon !A\otimes !B \to !(A\&B) \\ n^{-1} &\stackrel{\text{def}}{=} d_{A\&B}; !\mathsf{fst}\otimes !\mathsf{snd}\colon !(A\&B) \to !A\otimes !B \end{aligned}$$

$$\begin{aligned} p &\stackrel{\text{def}}{=} \mathsf{m}_I; !\top\colon I \to !\mathsf{t} \\ p^{-1} &\stackrel{\text{def}}{=} e_{\mathsf{t}}\colon !\mathsf{t} \to I \end{aligned}$$

Proof. By (long) construction. ■

Given these isomorphisms we can repeat the reasoning of Proposition 17 to get the following property of the co-Kleisli category associated with a Linear category.

Proposition 21. Given a Linear category, $\mathbb{C}$, its co-Kleisli category, $\mathbb{C}_!$, is cartesian closed.

It is important to note that the co-Kleisli category is only cartesian closed if $\mathbb{C}$ has (strong) products. If a Linear category does not have products it is still a model for the multiplicative, exponential fragment of **ILL**, unlike Seely's model which critically requires the products.

Given our earlier calculations we might consider the adjunction between a Linear category and its co-Kleisli category, where we find that the following holds.

Lemma 22. The adjunction between a Linear category, $\mathbb{C}$, (with finite products) and its co-Kleisli category, $\mathbb{C}_!$, is a *monoidal* adjunction.

Thus when considering the complete intuitionistic fragment, the new-Seely and Linear categories are equivalent. Theorem 25 shows that a monoidal adjunction between a particular SMCC (a new-Seely category) and CCC (its co-Kleisli category) yielded a structure of a Linear category. Lemma 22 shows that a Linear category has the structure of a monoidal adjunction between it (a SMCC) and its associated co-Kleisli category (a CCC). Thus the notion of a Linear category is in some senses equivalent to the existence of a monoidal adjunction between a SMCC and a CCC. This observation has been used by Benton [11] to derive the syntax of a mixed linear and non-linear term calculus.

7 Comparison: Lafont's Model

In his thesis [47, 46], Lafont proposed an alternative model for **ILL**. Rather than directly using comonad or coalgebra structures his model relies on imposing a strict condition involving a *comonoid* structure.

Definition 44. A *Lafont category*, $\mathbb{C}$ consists of

1. A SMCC with finite products
2. For each object A of $\mathbb{C}$, $!A$ is the co-free commutative comonoid co-generated by A

An alternative (and equivalent) definition is given by Lafont in terms of categorical combinators.

Definition 45. A Lafont category, $\mathbb{C}$, consists of

1. A SMCC with finite products
2. Three (families of) combinators:

$$\begin{array}{rcl} \mathsf{read}_A & : & !A \to A \\ \mathsf{kill}_A & : & !A \to I \\ \mathsf{dupl}_A & : & !A \to !A \otimes !A \end{array}$$

   such that $(!A, \mathsf{kill}_A, \mathsf{dupl}_A)$ forms a commutative comonoid.
3. A combinator formation rule

$$\frac{\mathsf{C}\colon A \to B \qquad \mathsf{er}\colon A \to I \qquad \mathsf{du}\colon A \to A \otimes A}{\mathsf{make}(\mathsf{C}, \mathsf{er}, \mathsf{du})\colon A \to !B}$$

   with the following conditions
   (a) $(A, \mathsf{er}, \mathsf{du})$ forms a commutative comonoid.
   (b) $\mathsf{make}(\mathsf{C}, \mathsf{er}, \mathsf{du})$ is the only morphism $\pi\colon A \to !B$ such that
      i. $\pi; \mathsf{read}_B = \mathsf{C}$
      ii. $\pi; \mathsf{kill}_B = \mathsf{er}$
      iii. $\pi; \mathsf{dupl}_B = \mathsf{du}$

Lafont showed that this could be considered a categorical model of **ILL** in a similar way to Seely: namely, by checking that the co-Kleisli category is cartesian closed.

Proposition 22. (Lafont) Given a Lafont category, $\mathbb{C}$, there are canonical isomorphisms

$$\begin{array}{rcl} !A \otimes !B & \cong & !(A \& B) \\ I & \cong & !\mathsf{t} \end{array}$$

Theorem 27. (Lafont) Given a Lafont category, $\mathbb{C}$, then $\mathbb{C}_!$ is cartesian closed.

Proof. By construction and use of Proposition 22. ■

Of course, as we have seen earlier with Seely's model, there is more to producing a sound model of **ILL** than simply ensuring that a particular construction yields a cartesian closed category. Rather than considering directly whether a Lafont category is a sound model of **ILL** we shall first give a slightly more abstract (but equivalent) definition of a Lafont category and then use this to consider the relationship between a Lafont category and a Linear category.

Definition 46. A *Lafont category* arises from requiring that there is the following adjunction between a SMCC $\mathbb{C}$, and its category of commutative comonoids, $\mathrm{coMon_c}(\mathbb{C})$.

$$\begin{array}{c} \mathrm{coMon_c}(\mathbb{C}) \\ F \downarrow \dashv \uparrow G \\ \mathbb{C} \end{array}$$

where G maps objects A to the 'free' comonoid (GA, d_A, e_A) and F is the forgetful functor (and G is *right* adjoint to F).

The definition of an adjunction tells us that there exists a natural transformation $\epsilon: FG \dot{\rightarrow} Id$, such that for all objects (A, d, e) in $\text{coMon}_c(\mathbb{C})$ and objects B in $\mathbb{C}$, for all morphisms $g: F(A, d, e) \rightarrow B$ there is a *unique* morphism $f: (A, d, e) \rightarrow G(B)$ such that the following diagram commutes:

$$\begin{array}{ccc} F(A,d,e) & \xrightarrow{Ff} & F(G(B)) \\ & \searrow^{g} & \downarrow{\scriptstyle \epsilon_B} \\ & & B \end{array}$$

Expanding out this diagram gives the following (where $! = FG$)

$$\begin{array}{ccc} A & \xrightarrow{f} & !B \\ & \searrow^{g} & \downarrow{\scriptstyle \epsilon_B} \\ & & B \end{array}$$

Since we have that f is a morphism in the category of comonoids, we know that it will also satisfy the following diagrams.

$$\begin{array}{ccc} A & \xrightarrow{f} & !B \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \mathsf{e}_B} \\ I & = & I \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & !B \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \mathsf{d}_B} \\ A \otimes A & \xrightarrow[f \otimes f]{} & !B \otimes !B \end{array}$$

Thus writing $\mathsf{make}(\mathsf{g}, d, e)$ for f and read for ϵ, we have precisely Lafont's (second) definition for a Lafont category. The definition of an adjunction also says that there exists a natural transformation $\eta: Id \dot{\rightarrow} GF$ such that for all objects (A, d, e) in $\text{coMon}_c(\mathbb{C})$ and objects B in $\mathbb{C}$, for all morphisms $f: (A, d, e) \rightarrow G(B)$ there is a unique morphism $g: F(A, d, e) \rightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} (A,d,e) & \xrightarrow{\eta_A} & GF(A,d,e) \\ & \searrow^{f} & \downarrow{\scriptstyle Gg} \\ & & G(B) \end{array}$$

It is worth pointing out that the components of the natural transformation η are morphisms in $\text{coMon}_c(\mathbb{C})$ and as such are *comonoid morphisms*.

We shall make use of the following fact that the forgetful functor $F: \text{coMon}_c(\mathbb{C}) \rightarrow \mathbb{C}$ preserves the symmetric monoidal structure 'on the nose'.

Lemma 23. The forgetful functor $F: \text{coMon}_c(\mathbb{C}) \rightarrow \mathbb{C}$ is a *strict* (and, hence, strong) symmetric monoidal functor $(F, m_{A,B}, m_I)$.

Given that we have an adjunction, basic category theory [53, Page 134] tells us that there is a comonad induced on $\mathbb{C}$.

Proposition 23. Given a Lafont category, there is a comonad $(FG, F\eta_G, \epsilon)$ induced on $\mathbb{C}$.

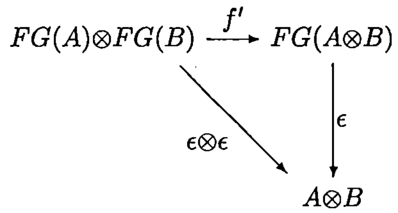
In fact, we can prove that this comonad is a monoidal comonad.

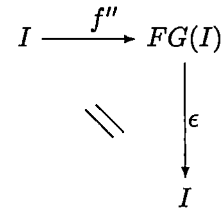
Proposition 24. Given a Lafont category, the induced comonad $(FG, F\eta_G, \epsilon)$ is a *monoidal* comonad $(FG, F\eta_G, \epsilon, \mathsf{m}_{A,B}, \mathsf{m}_I)$.

Proof. Take the following morphisms

$$\begin{aligned} \epsilon\otimes\epsilon &: FG(A)\otimes FG(B) \to A\otimes B \\ \mathsf{id}_I &: I \to I \end{aligned}$$

Then by definition of an adjunction we have that the following diagrams commute.

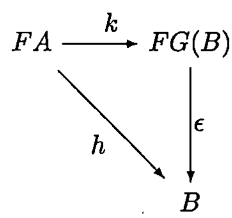




Moreover we know that the morphisms f' and f'' are unique and so we shall call them $\mathsf{m}_{A,B}$ and m_I respectively. It is worth pointing out that these morphisms are also comonoid morphisms. First we shall check the naturality of $\mathsf{m}_{A,B}$, given two morphisms $f: A \to C$ and $g: B \to D$, this implies the following required equality.

$$\mathsf{m}_{A,B}; FG(f\otimes g) \quad = \quad FG(f)\otimes FG(g); \mathsf{m}_{C,D}$$

By the definition of an adjunction, we know that for all morphisms $h: FA \to B$ there exists a unique morphism $k: FA \to FG(B)$ such that the following diagram commutes.



We shall take as h the morphism $(FG(f)\otimes FG(g); \epsilon\otimes\epsilon): FG(A)\otimes FG(B) \to C\otimes D$ and the two sides in the equality above as candidates for k. First we have to check that they satisfy the diagram. Considering the first candidate:

$$\begin{aligned} \mathsf{m}_{A,B}; FG(f\otimes g); \epsilon &= \mathsf{m}_{A,B}; \epsilon; f\otimes g \\ &= \epsilon\otimes\epsilon; f\otimes g \\ &= FG(f)\otimes FG(g); \epsilon\otimes\epsilon \end{aligned}$$

Considering the second candidate:

$$FG(f)\otimes FG(g); \mathsf{m}_{C,D}; \epsilon \quad = \quad FG(f)\otimes FG(g); \epsilon\otimes\epsilon$$

We can see that they are both valid candidates for k, but as k is unique by definition, we conclude that the candidates are equal, and thus the naturality equation holds.

Now we shall check that FG is a symmetric monoidal functor. Recalling Definition 25, this amounts to requiring four equalities to hold and we shall take each in turn. First consider the equality for λ.

$$\mathsf{m}_I\otimes\mathsf{id}_{FG(A)}; \mathsf{m}_{I,A}; FG(\lambda) \quad = \quad \lambda_{FG(A)}$$

Again, we verify this using the definition of an adjunction. We take as h the morphism $(\lambda_{FG(A)}; \epsilon): I\otimes FG(A) \to A$ and the two sides in the above equality as candidates for k. Checking the first candidate:

$$\begin{aligned}
\mathsf{m}_I \otimes \mathsf{id}; \mathsf{m}; FG(\lambda); \epsilon &= \mathsf{m}_I \otimes \mathsf{id}; \mathsf{m}; \epsilon; \lambda \\
&= \mathsf{m}_I \otimes \mathsf{id}; \epsilon \otimes \epsilon; \lambda \\
&= \mathsf{m}_I \otimes \mathsf{id}; \epsilon \otimes \mathsf{id}; \mathsf{id} \otimes \epsilon; \lambda \\
&= \mathsf{id} \otimes \epsilon; \lambda \\
&= \lambda; \epsilon
\end{aligned}$$

Checking the second candidate is trivial. Thus as they are both valid candidates, and the uniqueness of the adjunction tells us that they must be equal. The equality for ρ holds similarly and is omitted. Next we consider the associativity isomorphism α.

$$\mathsf{m} \otimes \mathsf{id}; \mathsf{m}; FG(\alpha) = \alpha; \mathsf{id} \otimes \mathsf{m}; \mathsf{m}$$

Again we verify this using the definition of an adjunction. We take as h the morphism $((\epsilon \otimes \epsilon) \otimes \epsilon; \alpha) \colon (FG(A) \otimes FG(B)) \otimes FG(C) \to A \otimes (B \otimes C)$ and the two sides in the above equality as candidates for k. Checking the first candidate:

$$\begin{aligned}
\mathsf{m} \otimes \mathsf{id}; \mathsf{m}; FG(\alpha); \epsilon &= \mathsf{m} \otimes \mathsf{id}; \mathsf{m}; \epsilon; \alpha \\
&= \mathsf{m} \otimes \mathsf{id}; \epsilon \otimes \epsilon; \alpha \\
&= \mathsf{m} \otimes \mathsf{id}; \epsilon \otimes \mathsf{id}; \mathsf{id} \otimes \epsilon; \alpha \\
&= (\epsilon \otimes \epsilon) \otimes \mathsf{id}; \mathsf{id} \otimes \epsilon; \alpha \\
&= (\epsilon \otimes \epsilon) \otimes \epsilon; \alpha
\end{aligned}$$

Checking the second candidate:

$$\begin{aligned}
\alpha; \mathsf{id} \otimes \mathsf{m}; \mathsf{m}; \epsilon &= \alpha; \mathsf{id} \otimes \mathsf{m}; \epsilon \otimes \epsilon \\
&= \alpha; \mathsf{id} \otimes \mathsf{m}; \mathsf{id} \otimes \epsilon; \epsilon \otimes \mathsf{id} \\
&= \alpha; \mathsf{id} \otimes (\epsilon \otimes \epsilon); \epsilon \otimes \mathsf{id} \\
&= \alpha; \epsilon \otimes (\epsilon \otimes \epsilon) \\
&= (\epsilon \otimes \epsilon) \otimes \epsilon; \alpha
\end{aligned}$$

Thus they are both valid candidates and the uniqueness of the adjunction tells us that they must be equal. Finally we consider the equality for the symmetry isomorphism γ.

$$\mathsf{m}_{A,B}; FG(\gamma) = \gamma; \mathsf{m}_{B,A}$$

Again we verify this using the definition of an adjunction. We take h to be the morphism $(\epsilon \otimes \epsilon; \gamma) \colon FG(A) \otimes FG(B) \to B \otimes A$ and the two sides in the above equality as candidates for k. Checking the first candidate:

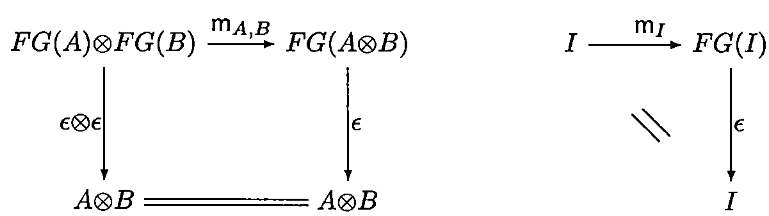
$$\begin{aligned}
\mathsf{m}; FG(\gamma); \epsilon &= \mathsf{m}; \epsilon; \gamma \\
&= \epsilon \otimes \epsilon; \gamma
\end{aligned}$$

Checking the second candidate:

$$\begin{aligned}
\gamma; \mathsf{m}; \epsilon &= \gamma; \epsilon \otimes \epsilon \\
&= \epsilon \otimes \epsilon; \gamma
\end{aligned}$$

Thus they are both valid candidates and the uniqueness of the adjunction tells us that they must be equal.

Now we have to check that ϵ is a monoidal natural transformation. Recalling Definition 27 this amounts to requiring that two diagrams commute. These diagrams are as follows.



Both these diagrams commute trivially by definition of an adjunction. Finally we shall check that $F\eta_G$ is a monoidal natural transformation which amounts to two equalities. Firstly, we require the equality:

$$\mathsf{m}_{A,B}; F\eta_G \quad = \quad F\eta_G \otimes F\eta_G; \mathsf{m}_{FG(A),FG(B)}; FG(\mathsf{m}_{A,B})$$

We shall verify this using the definition of an adjunction and take $\mathsf{m}_{A,B}$ as h, and the two sides in the above equality as candidates for k. Checking the first candidate:

$$\mathsf{m}; F\eta_G; \epsilon \quad = \quad \mathsf{m}$$

Checking the second candidate:

$$\begin{aligned} F\eta_G \otimes F\eta_G; \mathsf{m}; FG(\mathsf{m}); \epsilon &= F\eta_G \otimes F\eta_G; \mathsf{m}; \epsilon; \mathsf{m} \\ &= F\eta_G \otimes F\eta_G; \epsilon \otimes \epsilon; \mathsf{m} \\ &= \mathsf{m} \end{aligned}$$

Thus they are both valid candidates and the uniqueness of the adjunction tells us that they must be equal. The second equality is:

$$\mathsf{m}_I; F\eta_G \quad = \quad \mathsf{m}_I; FG(\mathsf{m}_I)$$

We verify this using the definition of an adjunction and take m_I for h and the two sides in the above equality as candidates for k. Checking the first candidate:

$$\mathsf{m}_I; F\eta_G; \epsilon \quad = \quad \mathsf{m}_I$$

Checking the second candidate:

$$\begin{aligned} \mathsf{m}_I; FG(\mathsf{m}_I); \epsilon &= \mathsf{m}_I; \epsilon; \mathsf{m}_I \\ &= \mathsf{m}_I \end{aligned}$$

Thus they are both valid candidates and the uniqueness of the adjunction tells us that they are equal. Thus we are done. ■

Let us briefly consider the comonoid structures. We know that G maps an object A to the 'free' comonoid (GA, d_A, e_A). Let us consider the structure $(FG(A), \mathsf{d}_A, \mathsf{e}_A)$ where we have the following definitions.

$$\begin{aligned} \mathsf{d}_A &\stackrel{\text{def}}{=} F(d_A); m_{GA,GA} \\ \mathsf{e}_A &\stackrel{\text{def}}{=} F(e_A); m_I \end{aligned}$$

We have from Lemma 23 that F is a strict, symmetric monoidal functor, and so both $m_{A,B}$ and m_I are identities. It is then easy to see the following.

Lemma 24. The structure $(FG(A), \mathsf{d}_A, \mathsf{e}_A)$ is a commutative comonoid.

Proof. By construction (Omitted). ■

Lemma 25. Given a Lafont category, $\mathbb{C}$, it is the case that $\mathsf{e}_A: FG(A) \to I$ and $\mathsf{d}_A: FG(A) \to FG(A) \otimes FG(A)$ are monoidal natural transformations.

Proof. The fact that they are monoidal is given by the fact that both $\mathsf{m}_{A,B}$ and m_I are monoidal morphisms by definition. For d to be a natural transformation, the following diagram must commute.

$$\begin{array}{ccc}
FG(A) & \xrightarrow{FG(f)} & FG(B) \\
{\scriptstyle F(d_A)}\downarrow & & \downarrow{\scriptstyle F(d_B)} \\
F(GA{\otimes}GA) & \xrightarrow{F(Gf{\otimes}Gf)} & F(GB{\otimes}GB) \\
{\scriptstyle m^{-1}}\downarrow & & \downarrow{\scriptstyle m^{-1}} \\
FG(A){\otimes}FG(A) & \xrightarrow[FG(f){\otimes}FG(f)]{} & FG(B){\otimes}FG(B)
\end{array}$$

The upper square commutes since Gf is a comonoid morphism for all morphisms f. The lower square commutes by naturality of m. For e to be a natural transformation the following diagram must commute.

$$\begin{array}{ccc}
FG(A) & \xrightarrow{FG(f)} & FG(B) \\
{\scriptstyle F(e_A)}\downarrow & & \downarrow{\scriptstyle F(e_B)} \\
FI & = & FI \\
{\scriptstyle m_I^{-1}}\downarrow & & \downarrow{\scriptstyle m_I^{-1}} \\
I & = & I
\end{array}$$

The upper square commutes since Gf is a comonoid morphism for all morphisms f. The lower square commutes trivially. ■

Lemma 26. Given a Lafont category, $\mathbb{C}$, it is the case that $\mathsf{e}_A: FG(A) \to I$ and $\mathsf{d}_A: FG(A) \to FG(A){\otimes}FG(A)$ are also coalgebra morphisms.

Proof. Requiring that $\mathsf{e}_A: FG(A) \to I$ is a coalgebra morphism amounts to the following equality.

$$\mathsf{e}_A; \mathsf{m}_I \quad = \quad F\eta_{GA}; FG(\mathsf{e}_A)$$

As in the previous proof, we shall verify this using the definition of an adjunction and take e_A as h and the two sides of the above equality as candidates for k. Checking the first candidate:

$$\mathsf{e}_A; \mathsf{m}_I; \epsilon \quad = \quad \mathsf{e}_A$$

Checking the second candidate:

$$\begin{aligned} F\eta_G; FG(\mathsf{e}_A); \epsilon &= F\eta_G; \epsilon; \mathsf{e}_A \\ &= \mathsf{e}_A \end{aligned}$$

Thus they are both valid candidates and the uniqueness of the adjunction tells us that they are equal.

Requiring that $\mathsf{d}_A: FG(A) \to FG(A){\otimes}FG(A)$ is a coalgebra morphism amounts to the following equality.

$$F\eta_{GA}; FG(\mathsf{d}_A) \quad = \quad \mathsf{d}_A; F\eta_{GA}\otimes F\eta_{GA}; \mathsf{m}_{FG(A),FG(A)}$$

We verify this using the definition of an adjunction and take d_A as h and the two morphisms in the above equality as candidates for k. Checking the first candidate:

$$\begin{aligned} F\eta_G; FG(\mathsf{d}); \epsilon &= F\eta_G; \epsilon; \mathsf{d} \\ &= \mathsf{d} \end{aligned}$$

Checking the second candidate:

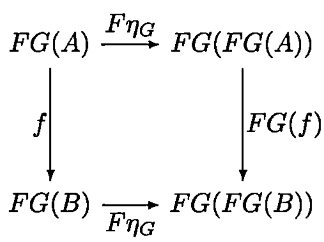
$$\begin{aligned} \mathsf{d}; F\eta_G\otimes F\eta_G; \mathsf{m}; \epsilon &= \mathsf{d}; F\eta_G\otimes F\eta_G; \epsilon\otimes\epsilon \\ &= \mathsf{d} \end{aligned}$$

Thus they are both valid candidates and the uniqueness of the adjunction tells us that they are equal.

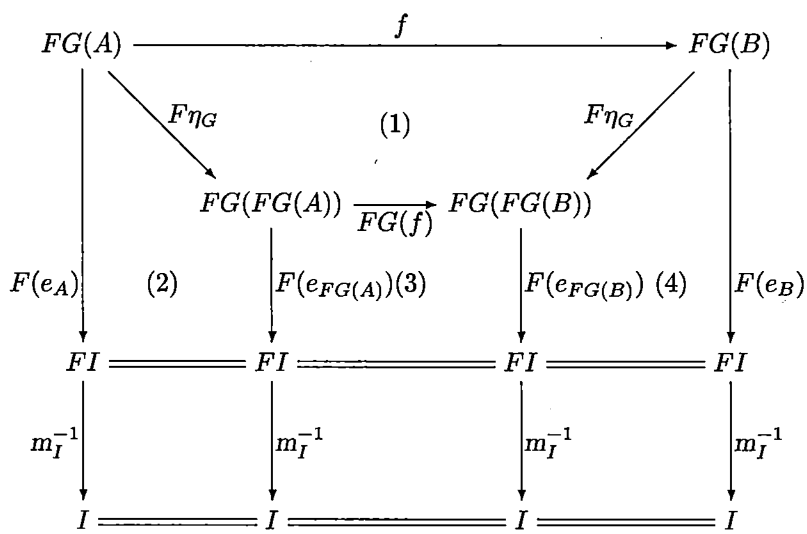
∎

Lemma 27. Given a Lafont category, $\mathbb{C}$, if $f\colon (FG(A), F\eta_G) \to (FG(B), F\eta_G)$ is a coalgebra morphism, then it is also a comonoid morphism.

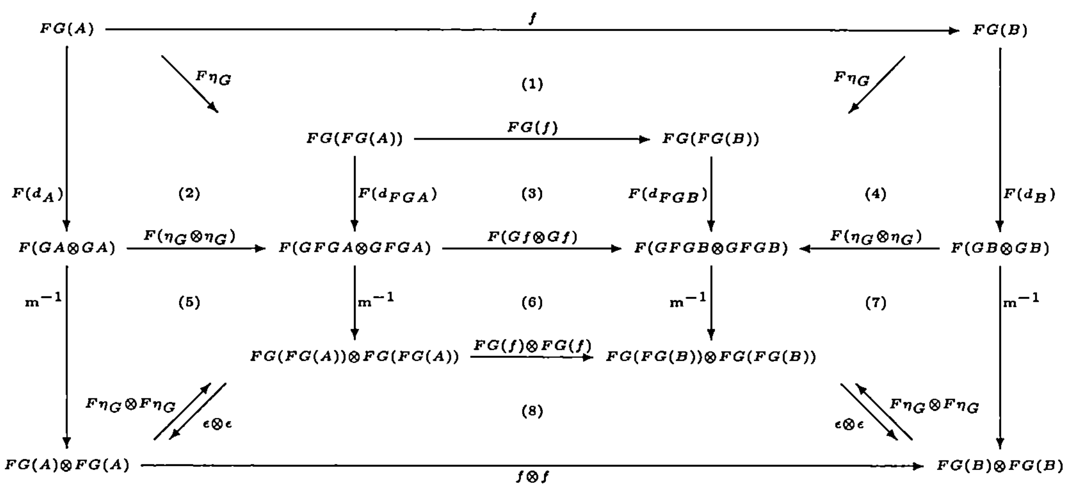
Proof. Assuming that f is a coalgebra morphism amounts to the following diagram commuting.



If f is a comonoid morphism then two diagrams must commute. Firstly:



Square (1) commutes by assumption. Squares (2) and (4) commute by the fact that η is a comonoid morphism. Square (3) commutes by the fact that Gf is a comonoid morphism for all morphisms f. The lower squares all commute trivially. The second diagram is as follows.



Square (1) commutes by assumption. Squares (2) and (4) commute by the fact that η is a comonoid morphism. Square (3) commutes by the fact that $G(f)$ is a comonoid morphism for all morphisms f. Squares (5), (6) and (7) commute by naturality of m^{-1} and finally square (8) commutes by naturality of ϵ. ■

Thus we can now state precisely the relationship between a Lafont category and a Linear category.

Theorem 28. Every Lafont category is a Linear category.

Proof. By Proposition 24 and Lemmas 25, 26 and 27. ■

We can show that a Lafont category is a sound model for the LTC-theory.

Theorem 29. A Lafont category, $\mathbb{C}$, is a model for the LTC-theory $(\mathcal{L}, \mathcal{A})$.

8 Translations

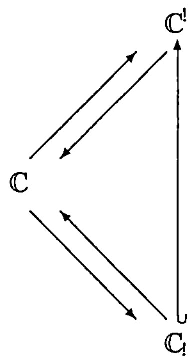
In Chapter 2 we saw how the exponential connective (!) via the Girard translation gives **ILL** the logical strength of **IL**. It is well known (see, for example, the book by Lambek and Scott [52]) that a categorical model of **IL** is a *cartesian closed category* (CCC). As we have modelled the exponential as a comonad (with particular structure) we would therefore expect this to generate a CCC in some way. In fact we have already seen how the comonad along with the so-called Seely isomorphisms in a Linear category ensures that the co-Kleisli construction yields a CCC. For completeness let us repeat that proposition.

Proposition 25. Given a Linear category, $\mathbb{C}$, the co-Kleisli category, $\mathbb{C}_!$, is cartesian closed.

Basic category theory [52, Corollary 6.9] shows that the co-Kleisli category is related to the category of coalgebras in the following way.

Lemma 28. The co-Kleisli category is equivalent to the full subcategory of the category of coalgebras consisting of all free coalgebras.

Thus we have the following situation, where $\mathbb{C}$ is a Linear category.



We shall consider the category of coalgebras in more detail, as well as a certain subcategory which was first identified by Hyland [15].

Proposition 26. Given a monoidal comonad, $(!, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_I)$, on a SMC $\mathbb{C}$, then $\mathbb{C}^!$ is a symmetric monoidal category.

Proof. The unit of this SMC is the coalgebra (I, m_I). Given two coalgebras (A, h_A) and (B, h_B) their tensor product is defined to be

$$(A, h_A) \otimes^! (B, h_B) \stackrel{\text{def}}{=} (A \otimes B, (h_A \otimes h_B; \mathsf{m}_{A,B})).$$

It is easy to verify that this is a coalgebra. Checking that the category of coalgebras is a SMC is relatively simple and omitted. ■

It is interesting to note in the above proof the essential use of the monoidal structure of the comonad and also the fact that given two free coalgebras $(!A, \delta_A)$ and $(!B, \delta_B)$, their tensor product $(!A \otimes !B, (\delta \otimes \delta; \mathsf{m}_{A,B}))$ is *not* necessarily a free coalgebra.

The next obvious question is whether a symmetric monoidal *closed* category induces a closed structure on the category of coalgebras. We find that this is not quite the case although we can isolate those cases where it is true.

Proposition 27. Given a monoidal comonad $(!, \varepsilon, \delta, \mathsf{m}_{A,B}, \mathsf{m}_I)$ on a SMCC $\mathbb{C}$, then within the category of coalgebras, $\mathbb{C}^!$, the free coalgebras $(!A, \delta)$ have an internal hom.

Proof. We shall show that

$$(B, h_B) \multimap^! (!A, \delta_A) \stackrel{\text{def}}{=} (!(B \multimap A), \delta_{B \multimap A}),$$

is an internal hom. This can be checked by simple analysis of the adjunction $F \dashv G$ between $\mathbb{C}^!$ and $\mathbb{C}$, where the forgetful functor F forgets the coalgebra structure, and the functor G maps any object A to the free coalgebra $(!A, \delta_A)$. By definition the following bijection holds.

$$\frac{(A, h_A) \xrightarrow{f} GB = (!B, \delta_B)}{\overline{\quad F(A, h_A) = A \xrightarrow{\hat{f}} B \quad}}$$

Hence the following sequence of bijections shows that the free coalgebras have an internal hom.

$$\begin{array}{c}
(C, h_C) \longrightarrow (!(B \multimap A), \delta) \qquad \text{in } \mathbb{C}^! \\
\hline\hline
C \longrightarrow B \multimap A \qquad \text{in } \mathbb{C} \\
\hline\hline
C \otimes B \longrightarrow A \qquad \text{in } \mathbb{C} \\
\hline\hline
(C \otimes B, (h_C \otimes h_B; \mathsf{m})) \longrightarrow (!A, \delta) \qquad \text{in } \mathbb{C}^!
\end{array}$$

■

We can now consider some properties of the category of coalgebras for a Linear category.

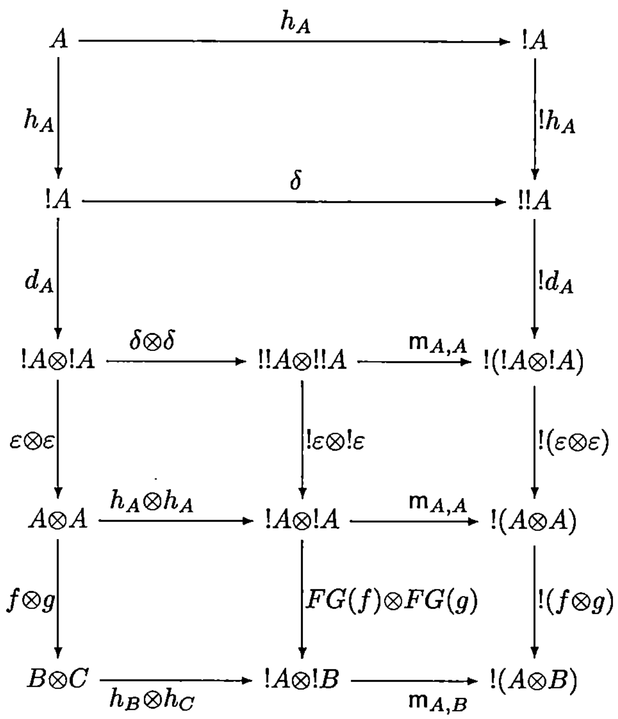
Proposition 28. Given a Linear category $\mathbb{C}$ and consider its category of coalgebras, $\mathbb{C}^!$. The tensor product of any two coalgebras (A, h_A) and (B, h_B) is actually a cartesian product.

Proof. We recall from Proposition 26 that the tensor product of the two coalgebras (A, h_A) and (B, h_B) is defined as $(A \otimes B, (h_A \otimes h_B; \mathsf{m}_{A,B}))$. Proposition 12 shows that in the category of coalgebras every coalgebra has a monoidal structure given by the two natural transformations $d^!$ and $e^!$. These natural transformations can be used to define projection and diagonal morphisms, *viz.*

$$\begin{array}{rcl} \mathsf{fst}^! & \stackrel{\text{def}}{=} & \mathsf{id}_A \otimes e^!; \rho_A: A \otimes B \to A \\ \mathsf{snd}^! & \stackrel{\text{def}}{=} & e^! \otimes \mathsf{id}_B; \lambda_A: A \otimes B \to B \\ \Delta^! & \stackrel{\text{def}}{=} & d^!: A \to A \otimes A. \end{array}$$

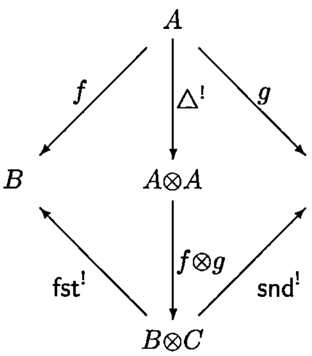
Propositions 13 and 15 show that $e^!$ and $d^!$ are coalgebra morphisms and it is trivial to see that the projection and diagonal are also coalgebra morphisms.

Let us check the proposed cartesian structure. Take a coalgebra $(A, h_A: A \to !A)$ such that $f: (A, h_A) \to (B, h_B)$ and $g: (A, h_A) \to (C, h_C)$ are coalgebra morphisms. We can take as a morphism from (A, h_A) to $(B \otimes C, (h_B \otimes h_C; \mathsf{m}_{B,C}))$ the composite $\Delta^!; f \otimes g$. For this composite to be a coalgebra morphism the following diagram must commute (recalling that $d^!$ is defined as the composite $h_A; d_A; \varepsilon \otimes \varepsilon$).

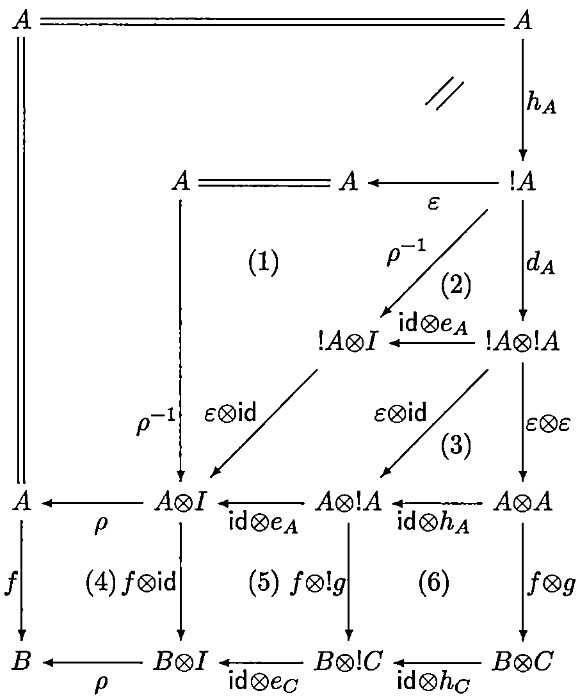


The upper square commutes by definition of h_A being a coalgebra structure map. The upper middle square commutes as d is a coalgebra morphism. The left lower middle square commutes by Proposition 14 (the fact that $(h_A; d_A)$ is a cofork of $\delta \otimes \delta; !\varepsilon \otimes !\varepsilon$ and $\varepsilon \otimes \varepsilon; h_A \otimes h_A$). The right lower middle square commutes by naturality of m. The left lower square commutes as f and g are coalgebra morphisms by assumption and the right lower square commutes by naturality of m.

For there to be a cartesian structure we require the following diagram to commute.



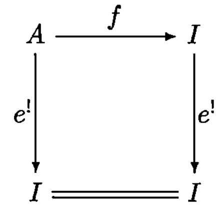
We shall consider only the left hand square as the right hand one is similar. Expanding out the definitions of $\Delta^!$ and $\mathsf{fst}^!$, gives the following diagram



Square (1) commutes by naturality of ρ^{-1}. Triangle (2) commutes by the definition of a commutative comonoid (Definition 31). Triangle (3) does *not* commute but holds from an earlier cofork equality (Proposition 11). Square (4) commutes by naturality of ρ, square (5) by naturality of e and square (6) by g being a coalgebra morphism. The other squares commute trivially. ■

Lemma 29. Given a Linear category $\mathbb{C}$ and consider its category of coalgebras, $\mathbb{C}^!$. The coalgebra (I, m_I) is the terminal object and the morphism $e^!: A \to I$ is the terminal morphism.

Proof. Recall that $e^! \stackrel{\text{def}}{=} h_A; e_A$. Assuming that there is another terminal morphism $f: A \to I$, which as it is a coalgebra morphism must also be a comonoid morphism, and so make the following diagram commute.



Thus the following holds.

$$\begin{array}{rcll} e^! & = & f; e^! & \\ & = & f; \mathsf{m}_I; e & \text{Def} \\ & = & f; \mathsf{id} & e \text{ is monoidal} \\ & = & f & \end{array}$$

■

Proposition 27 shows that internal homs exists for all the free coalgebras. Proposition 28 shows that in the category of coalgebras for a linear category the tensor product is actually a cartesian product. Thus all the free coalgebras are *exponentiable*, i.e. have internal homs with respect to the cartesian structure. It is not just the free coalgebras which are exponentiable. Take two exponentiable coalgebras (C, h_C) and (D, h_D) and form their product (Proposition 28 gives that the product is equivalent to the tensor product). We take as the internal hom of the product of these coalgebras

$$(B, h_B) \multimap^! ((C, h_C) \&^! (D, h_D)) \stackrel{\text{def}}{=} ((B, h_B) \multimap^! (C, h_C)) \&^! ((B, h_B) \multimap^! (D, h_D)),$$

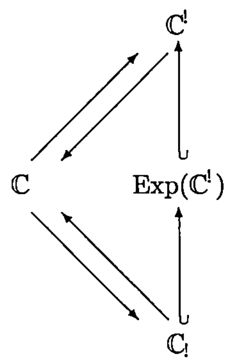
for all coalgebras (B, h_B). Then the product of two exponentiable coalgebras is also an exponentiable coalgebra by the following reasoning.

$$\begin{array}{c} \mathbb{C}^! (A \otimes B, C \& D) \\ \hline\hline \mathbb{C}^! (A \otimes B, C) \& \mathbb{C}^! (A \otimes B, D) \\ \hline\hline \mathbb{C}^! (A, B \multimap C) \& \mathbb{C}^! (A, B \multimap D) \\ \hline\hline \mathbb{C}^! (A, (B \multimap C) \& (B \multimap D)) \\ \hline\hline \mathbb{C}^! (A, B \multimap C \& D) \end{array}$$

The following subcategory was first identified by Hyland [15].

Theorem 30. (Hyland) The full subcategory of exponentiable objects, $\mathrm{Exp}(\mathbb{C}^!)$, of the category of coalgebras of a Linear category $\mathbb{C}$ forms a CCC containing the category of free coalgebras.

Thus we have the following situation where $\mathbb{C}$ is a Linear category.



We have considered two particular subcategories of the category of coalgebras which are cartesian closed, a natural question is to ask which conditions are necessary to make the whole category cartesian closed.

Definition 47. A pair of morphisms $f, g\colon A \to B$ is said to be a *coreflexive* pair if there exists a morphism $k\colon B \to A$ such that $f; k = \mathsf{id}_A = g; k$.

Proposition 29. In the category of coalgebras for a Linear category, $\mathbb{C}^!$, the morphism h_C is an *equalizer* of the (coreflexive) pair $(!h_C, \delta_C)$, i.e.

$$(C, h_C) \xrightarrow{h_C} (!C, \delta_C) \underset{\delta_C}{\overset{!h_C}{\rightrightarrows}} (!!C, \delta_{!C})$$

Proof. The pair $(!h_C; \delta_C)$ is a coreflexive pair as $!h_C; !\varepsilon_C = \mathsf{id}_{!C} = \delta_C; !\varepsilon_C$. The morphism h_C is a cofork of the pair (h_C, δ) by definition of a coalgebra. Assume that there exists another coalgebra morphism $f: (B, h_B) \to (!C, \delta_C)$ such that $f; !h_C = f; \delta_C$. Hence the following holds.

$$\begin{array}{rcl} f; \delta_C & = & f; !h_C \\ f; \delta_C; \varepsilon_{!C} & = & f; !h_C; \varepsilon_{!C} \\ f & = & f; !h_C; \varepsilon_{!C} \\ f & = & (f; \varepsilon_C); h \end{array}$$

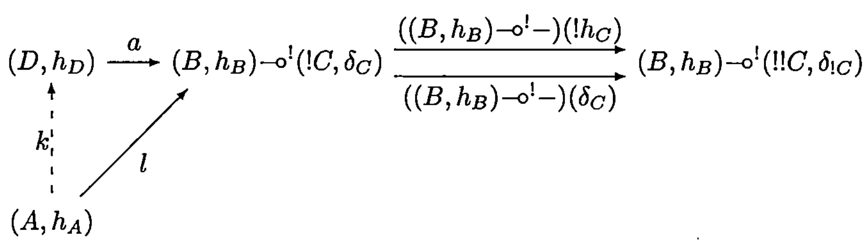
Thus given any f which is also a cofork of the pair $(!h, \delta)$, there is a morphism, $k(= f; \varepsilon_C)$, such that $k; h_C = f$. It is easy to see that k is unique and it is a coalgebra morphism by the following reasoning.

$$\begin{array}{rcl} (f; \varepsilon_C); h_C & = & f; !h_C; \varepsilon_{!C} \\ & = & f; \delta_C; \varepsilon_{!C} \\ & = & f \\ & = & f; \delta_C; !\varepsilon_C \\ & = & h_B; !f; !\varepsilon_C \\ & = & h_B; !(f; \varepsilon_C) \end{array}$$

■

Theorem 31. (Hyland) Given a Linear category $\mathbb{C}$ and consider its category of coalgebras $\mathbb{C}^!$. This category is cartesian closed if it has equalizers of coreflexive pairs.

Proof. (Sketch) The problematic part of this theorem is showing that an internal hom exists between any two coalgebras. Recalling from Lemma 27 that internal homs exist for all free coalgebras (where $(B, h_B) \multimap^! (!C, \delta_C) \stackrel{\text{def}}{=} (!(B \multimap C), \delta_{B \multimap C}))$ and taking the coreflexive pair from Proposition 29, there is the following equalizer diagram.



Taking a coalgebra morphism $f: A \otimes B \to C$, we form the morphism $(h_A; !\mathsf{Cur}(f)): A \to (B, h_B) \multimap^! (!C, \delta_C)$. It is simple to see that this is a cofork for the coreflexive pair $(((B, h_B) \multimap^! -)(!h_C), ((B, h_B) \multimap^! -)(\delta_C))$. Then by the equalizer assumption there is a unique morphism $k: (A, h_A) \to (D, h_D)$.

In the opposite direction, taking a coalgebra morphism $g: (A, h_A) \to (D, h_D)$, one can form the morphism $(g \otimes \mathsf{id}_B; a \otimes \mathsf{id}_B; \mathsf{id}_{!(B \multimap C)} \otimes h_B; \mathsf{m}; !\mathsf{App}): A \otimes B \to !C$. It is routine to see that this is a cofork for the (coreflexive) pair $(!h_C, \delta_C)$. From Proposition 29 there is then a unique coalgebra morphism $j: A \otimes B \to C$.

It is routine to see that the two operations described above form a bijection, i.e.

$$\frac{f: (A \otimes B) \to C}{\hat{f}: A \to D}.$$

Hence (D, h_D) can be taken as the internal hom $(B, h_B) \multimap^! (C, h_C)$. ■

Jacobs [45] has also considered this construction but he takes a different candidate for the internal hom which requires the stronger condition that there are equalizers for *all* parallel pairs of morphisms in $\mathbb{C}^!$.

We shall now consider the rôle of coproducts. We have assumed that a Linear category, $\mathbb{C}$, has coproducts and so we shall consider whether the three candidate CCCs have a coproduct structure induced on them.

Seely [69] demonstrated that the co-Kleisli category $\mathbb{C}_!$ does *not* have coproducts if $\mathbb{C}$ does. He pointed out that there are the following bijections.

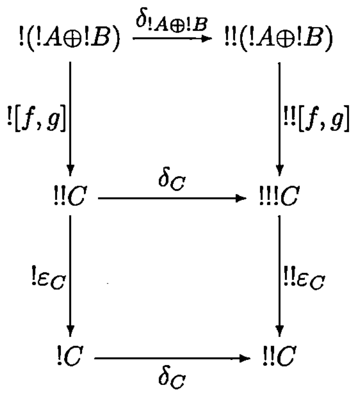
$$\dfrac{\dfrac{\mathbb{C}_!(A, C)}{\mathbb{C}(!A, C)} \qquad \dfrac{\mathbb{C}_!(B, C)}{\mathbb{C}(!B, C)}}{\mathbb{C}(!A \oplus !B, C)}$$

We might be tempted to insist that $!(A \oplus B) \cong !A \oplus !B$ (i.e. make the coproduct of two free coalgebras isomorphic to a free coalgebra) which would certainly entail that the co-Kleisli category has coproducts but as it is *not* logically the case that $!(A \oplus B) \vdash !A \oplus !B$, this would seem to be too strong a condition.

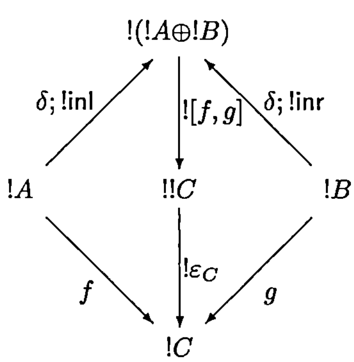
However it *is* possible to identify a *weak* coproduct structure in $\mathbb{C}_!$.

Lemma 30. Given two free coalgebras $(!A, \delta_A)$ and $(!B, \delta_B)$, we define their coproduct to be $(!(!A \oplus !B), \delta_{!A \oplus !B})$. We define the injection morphisms to be $\mathsf{inl}^! \stackrel{\text{def}}{=} \delta_A; !\mathsf{inl}\colon !A \to !(!A \oplus !B)$ and $\mathsf{inr}^! \stackrel{\text{def}}{=} \delta_B; !\mathsf{inr}\colon !B \to !(!A \oplus !B)$, which are coalgebra morphisms. Given two (free) coalgebra morphisms $f\colon !A \to !C$ and $g\colon !B \to !C$, then the morphism $(![f, g]; !\varepsilon_C)\colon !(!A \oplus !B) \to !C$ is a (free) coalgebra morphism and makes a coproduct diagram commute.

Proof. That the morphism $(![f, g]; !\varepsilon_C)$ is a coalgebra morphism amounts to the following diagram.



Both squares commute by naturality of δ. The fact that the injection morphisms are coalgebra morphisms is trivial and omitted. The relevant coproduct diagram is of the form:



Let us consider the left hand triangle. This commutes by the following equational reasoning.

$$\begin{array}{rcll} \delta; !\mathsf{inl}; ![f,g]; !\varepsilon & = & \delta; !f; !\varepsilon & \\ & = & f; \delta; !\varepsilon & f \text{ is a coalgebra morphism} \\ & = & f & \end{array}$$

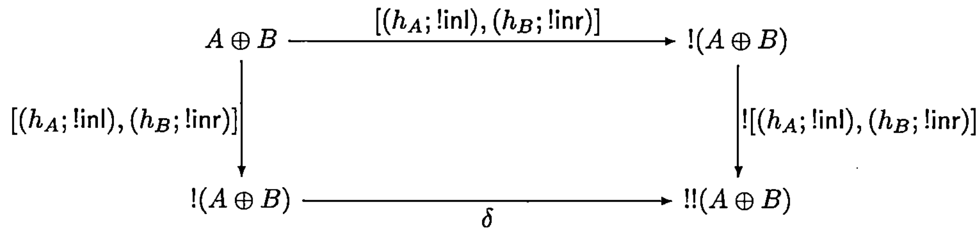
∎

The rôle of coproducts in Hyland's subcategory is slightly more problematic. It is *not* the case that it has coproducts, nor weak coproducts. This would seem to suggest that a further subcategory of coalgebras could be defined, which consists of those coalgebras which not only have internal homs with respect to the cartesian structure, but also have a (possibly weak) coproduct structure. This remains future work.

Let us consider the coproducts in the category of coalgebras.

Lemma 31. If (A, h_A) and (B, h_B) are coalgebras, then $(A \oplus B, [(h_A; !\mathsf{inl}), (h_B; !\mathsf{inr})])$ is a coalgebra.

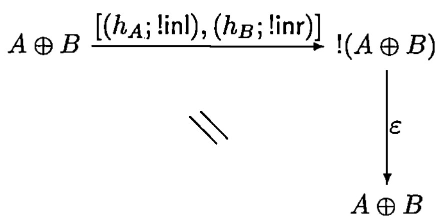
Proof. We require two diagrams to commute. The first diagram is as follows.



This diagram commutes by the following equational reasoning.

$$\begin{array}{l} [(h_A; !\mathsf{inl}), (h_B; !\mathsf{inr})]; ![(h_A; !\mathsf{inl}), (h_B; !\mathsf{inr})] \\ \quad = \ [(h_A; !\mathsf{inl}; ![(h_A; !\mathsf{inl}), (h_B; !\mathsf{inr})]), (h_B; !\mathsf{inr}; ![(h_A; !\mathsf{inl}), (h_B; !\mathsf{inr})])] \\ \quad = \ [(h_A; !(h_A; !\mathsf{inl})), (h_B; !(h_B; !\mathsf{inr}))] \\ \quad = \ [(h_A; !h_A; !!\mathsf{inl}), (h_B; !h_B; !!\mathsf{inr})] \\ \quad = \ [(h_A; \delta; !!\mathsf{inl}), (h_B; \delta; !!\mathsf{inr})] \\ \quad = \ [(h_A; !\mathsf{inl}; \delta), (h_B; !\mathsf{inr}; \delta)] \\ \quad = \ [(h_A; !\mathsf{inl}), (h_B; !\mathsf{inr})]; \delta \end{array}$$

The second diagram is as follows.



This diagram commutes by the following equational reasoning.

$$\begin{array}{rcl} [(h_A; !\mathsf{inl}), (h_B; !\mathsf{inr})]; \varepsilon & = & [(h_A; !\mathsf{inl}; \varepsilon), (h_B; !\mathsf{inr}; \varepsilon)] \\ & = & [(h_A; \varepsilon; \mathsf{inl}), (h_B; \varepsilon; \mathsf{inr})] \\ & = & [\mathsf{inl}, \mathsf{inr}] \\ & = & \mathsf{id}_{A \oplus B} \end{array}$$

∎

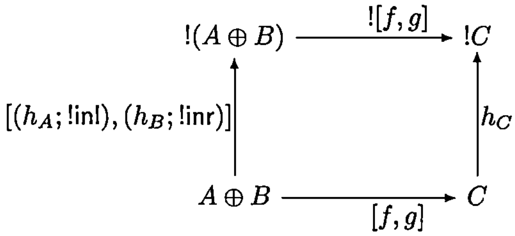
We can easily see that the following is true.

Lemma 32. The morphisms $\mathsf{inl}: A \to A \oplus B$ and $\mathsf{inr}: B \to A \oplus B$ are coalgebra morphisms.

Let us now consider the morphism $[f,g]$ where f and g are coalgebra morphisms.

Lemma 33. Given two coalgebra morphisms $f\colon A \to C$ and $g\colon B \to C$, then $[f,g]\colon A \oplus B \to C$ is a coalgebra morphism.

Proof. We require the following diagram to commute.



We shall prove this equationally.

$$\begin{array}{rcl} [(h_A;!\mathsf{inl}),(h_B;!\mathsf{inr})];![f,g] & = & [(h_A;!\mathsf{inl};![f,g]),(h_B;!\mathsf{inr};![f,g])] \\ & = & [(h_A;!f),(h_B;!g)] \\ & = & [(f;h_C),(g;h_C)] \\ & = & [f,g];h_C \end{array}$$

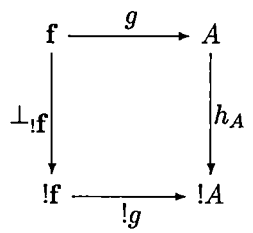
■

In fact we can identify the initial object and initial morphism.

Lemma 34. Given a SMC, $\mathbb{C}$, with a monoidal comonad and coproducts, then $(\mathbf{f},\perp_{!\mathbf{f}})$ is a coalgebra.

Lemma 35. Given a Linear category $\mathbb{C}$ and consider its category of coalgebras, $\mathbb{C}^!$. The coalgebra $(\mathbf{f},\perp_{!\mathbf{f}})$ is the initial object and the morphism $\perp_A\colon \mathbf{f} \to A$ is the initial morphism.

Proof. Assume that we have another morphism $g_A\colon \mathbf{f} \to A$. As it is a coalgebra morphism the following diagram must commute.



Then we have

$$\begin{array}{rcl} g;h_A & = & \perp_{!\mathbf{f}};!g \\ g;h_A & = & \perp_{!A} \\ g;h_A;\varepsilon_A & = & \perp_{!A};\varepsilon_A \\ g & = & \perp_A \end{array}$$

■

Proposition 30. Given a Linear category $\mathbb{C}$ (with coproducts) then the category of coalgebras $\mathbb{C}^!$ has coproducts.

Chapter 5
Conclusions and Further Work

In this chapter we summarize the work presented in this thesis. We then present and compare an alternative natural deduction formulation proposed by Troelstra [76]. We also briefly introduce classical linear logic (**CLL**) and consider some of its features. We then conclude by considering some immediate further work arising from this thesis.

1 Summary of Thesis

In Chapter 2 we presented the proof theory of **ILL**. We took the sequent calculus of Girard and gave a detailed proof of cut elimination. We then saw how this implied a simple proof of the subformula property. We then derived a natural deduction formulation, observing that our formulation was closed under substitution. We also saw how we have a number of choices with regards to the additive connectives. We then gave the β-reductions. The subformula property was considered and shown to hold provided a number of new reductions, the commuting conversions, were added. We then showed how an axiomatic formulation can be derived from the natural deduction formulation. Functions were given to map deductions from one formulation to another. Finally we considered the translation of Girard, which translates deductions in **IL** into deductions in **ILL**.

In Chapter 3 we saw how we could apply the Curry-Howard correspondence to the natural deduction formulation to derive a term assignment system for **ILL**. We also considered how to derive a term assignment system given a sequent calculus formulation. We derived a linear combinatory logic, by applying a Curry-Howard correspondence to the axiomatic formulation from the previous chapter. The various methods of proof reduction from Chapter 2 were analysed at the level of terms to suggest term reduction rules. We then considered the β-reduction rules and showed how the reduction system is both strongly normalizing and confluent. We also considered how the linear term calculus could be 'compiled' into linear combinators. Finally we gave Girard's translation at the level of terms.

In Chapter 4 we reviewed the terminology of equational logic. We took the term assignment system and the β-reduction rules as an equational theory and then analysed it categorically. This analysis not only resulted in suggesting certain categorical structures, but also in suggesting some new term equalities (some of which we had seen in Chapter 3). We then defined a model for **ILL**, a *Linear category*, and considered some of its properties. Alternative models by Seely and Lafont were then studied and compared. Surprisingly Seely's model was shown to be unsound as it does not model some term equalities with equal morphisms. We showed how Seely's model could be seen in a more abstract setting and how this suggested an alternative definition which lead to a sound model, which was shown to yield the structure of a Linear category. Lafont's model was studied and also shown to yield the structure of a Linear category. Finally we considered a categorical equivalent of Girard's translation using a construction due to Hyland.

2 An Alternative Natural Deduction Presentation

As we saw in Chapter 2, the natural deduction formulation of the *Promotion* rule is slightly unusual as it not only includes a check of all the free assumptions but also introduces many parasitic formulae, the consequence of which is the need for an extra commuting conversion. We might ask whether there is a simpler formulation. Prawitz [61] considered formulating certain modal logics in a natural deduction system. As we have remarked before, the rules for the necessity modality ($\Box$) are the same as *Promotion* and *Dereliction.* Troelstra [76] has subsequently taken Prawitz's ideas and applied them to **ILL**.

The idea for the alternative formulation of the *Promotion* rule is that we relax the restriction that all the open assumptions are of the form $!A_i$, to one which says that we can find a complete set of assumptions of the form $!A_i$ which could have subsequently had deductions substituted in for them.[1] In tree-form this amounts to the rule (where the complete set of assumptions are in bold face)

$$\begin{array}{c} \Delta_1 \qquad\quad \Delta_n \\ \vdots \qquad\quad\ \vdots \\ !\mathbf{A_1}\ \cdots\ !\mathbf{A_n} \\ \vdots \\ \dfrac{B}{!B}\ \textit{Promotion}'. \end{array}$$

In term form this rule is of the form

$$\frac{\Gamma_1,\ldots,\Gamma_n \rhd M[\vec{x} := \vec{N}]: B \qquad \text{where } \vec{x} = FV(M) \text{ and } \Gamma_i \rhd N_i : !A_i \text{ for some } A_i}{\Gamma_1,\ldots,\Gamma_n \rhd \mathsf{bang}(M[\vec{x} := \vec{N}]): !B}\ \textit{Promotion}'.$$

Troelstra has shown that there is an equivalence between this formulation and that presented in Chapter 2, in that there exist mappings from one to the other and vice versa. As Troelstra points out an immediate advantage of this formulation is that the commuting conversions previously required for the *Promotion* rule are *not* needed in this formulation (the conversion is mapped to an identity on derivations). However, returning to the term system there do appear to be problems. For example, consider the following derivation in this new formulation

$$\dfrac{\dfrac{\dfrac{\dfrac{!!!A}{(1)\quad !!A}\ \textit{Dereliction}}{(2)\quad !A}\ \textit{Dereliction}}{A}\ \textit{Dereliction}}{!A}\ \textit{Promotion}'.$$

The term assigned to this deduction is

$$x{:}\, !!!A \rhd \mathsf{bang}(\mathsf{derelict}(\mathsf{derelict}(\mathsf{derelict}(x)))): !A.$$

The problem is determining whether the assumption which has been substituted for is (1) or (2) in the above proof, or in other words, whether the above term is morally of the form

$$\mathsf{bang}(\mathsf{derelict}(\mathsf{derelict}(z)))[z := \mathsf{derelict}(x)]$$

or

$$\mathsf{bang}(\mathsf{derelict}(z))[z := \mathsf{derelict}(\mathsf{derelict}(x))].$$

Of course, in the formulation presented in this thesis, these alternatives represent two distinct derivations, *viz.*

$$\dfrac{\dfrac{!!!A}{!!A}\ \textit{Dereliction} \qquad \dfrac{\dfrac{[!!A]}{!A}\ \textit{Dereliction}}{A}\ \textit{Dereliction}}{!A}\ \textit{Promotion}$$

[1] This complete set of assumptions is called a *basis* by Troelstra [76].

and

$$\cfrac{\cfrac{\cfrac{!!!A}{!!A}\,\textit{Dereliction}}{!A}\,\textit{Dereliction} \qquad \cfrac{[!A]}{A}\,\textit{Dereliction}}{!A}\,\textit{Promotion},$$

which are assigned the (distinct) terms

$$x\colon !!!A \rhd \mathsf{promote}\ \mathsf{derelict}(x)\ \mathsf{for}\ z\ \mathsf{in}\ \mathsf{derelict}(\mathsf{derelict}(z))\colon !A$$

and

$$x\colon !!!A \rhd \mathsf{promote}\ \mathsf{derelict}(\mathsf{derelict}(x))\ \mathsf{for}\ z\ \mathsf{in}\ \mathsf{derelict}(z)\colon !A.$$

The alternative formulation essentially collapses these two derivations into one. Let us consider the consequences of this with respect to the categorical model. The two terms above are modelled by the morphisms

$$\varepsilon_{!!A}; \delta_{!A}; !\varepsilon_{!!A}; !\varepsilon_{!A}\colon !!!A \to !A$$

and

$$\varepsilon_{!!A}; \varepsilon_{!A}; \delta_{A}; !\varepsilon_{A}\colon !!!A \to !A,$$

respectively. If we insist that these morphisms are equal, we arrive at the equality

$$\varepsilon_{!!A}; !\varepsilon_{A} \quad = \quad \varepsilon_{!!A}; \varepsilon_{!A}.$$

Precomposing with the morphism $\delta_{!A}$ gives

$$!\varepsilon_{A} \quad = \quad \varepsilon_{!A}. \tag{5.1}$$

We have seen on page 131 that this equality is sufficient to make the comonad *idempotent*.[2]

A final problem is with the relationship between the alternative definitions of a normal form. One particular term equality (from Figure 4.5) which we derived from the categorical model is

$$\cfrac{}{x\colon !A \rhd \mathsf{promote}\ x\ \mathsf{for}\ y\ \mathsf{in}\ \mathsf{derelict}(y) = x\colon !A}\,\textit{Comonad}.$$

The corresponding term equality in the alternative formulation is

$$\cfrac{}{x\colon !A \rhd \mathsf{bang}(\mathsf{derelict}(x)) = x\colon !A}\,\textit{Comonad}'.$$

However, consider the term

$$x\colon !!A \rhd \mathsf{promote}\ \mathsf{derelict}(x)\ \mathsf{for}\ y\ \mathsf{in}\ y\colon !!A,$$

which is in normal form. This corresponds to the term

$$x\colon !!A \rhd \mathsf{bang}(\mathsf{derelict}(x))\colon !!A,$$

in the new formulation, which is *not* in normal form, as the *Comonad'* rule applies, and so it is rewritten to the term $x\colon !!A \rhd x\colon !!A$. (Again, analysing this categorically will lead to an idempotent comonad). So there appears to be a mismatch between the respective notions of normal form.

Thus, despite the proof theoretical advantage of this alternative formulation (the removal of the commuting conversions for the *Promotion* rule), we can identify immediately three problems with this alternative formulation with respect to the approach we have taken in this thesis.

---

[2]In a slightly different context, Wadler [79] has also pointed out how a similar syntax implies an idempotent comonad.

1. Terms do *not* uniquely encode deductions, but rather represent *sets* of deductions. Lincoln and Mitchell [55] have also considered a term calculus based on linear logic where this is the case.

2. A corresponding notion of normal form does not appear to be straightforward to define.

3. Categorically, the term syntax suggests an idempotent comonad. Nearly all models of **ILL** do *not* have this feature. (Filinski's proposal [28] is the only example of which I am aware which has an idempotent comonad.)

3 Classical Linear Logic

This thesis has concentrated solely on the intuitionistic fragment of linear logic. Of course, there is a classical version of linear logic (**CLL**). Here derivations are of the form $\Gamma \vdash \Delta$, i.e. where we have more than one conclusion. In addition there are some new connectives and units. For completeness we shall give the (two-sided) sequent calculus formulation in Figure 5.1.

The exciting feature of **CLL** is that it is a *constructive* logic and it has the logical power of classical logic. Thus if we extend the translation procedure from Chapter 2 by

$$(\neg A)^\circ \stackrel{\text{def}}{=} !A^\circ \multimap \mathbf{f},$$

then the equivalence can be formalized as follows.

Proposition 31. (Girard) If $\vdash_{CL} \Gamma \vdash \Delta$ then $\vdash_{CLL} !\Gamma^\circ \vdash \Delta^\circ$.

One particular feature of note is *linear negation* $(-)^\perp$. In particular it obeys the same kind of laws as classical negation. It can be characterized as follows.

Definition 48.

$$A \multimap B \stackrel{\text{def}}{=} A^\perp ⅋ B$$

$$\begin{array}{rcl}
(A^\perp)^\perp & \equiv & A \\
(A \otimes B)^\perp & \equiv & A^\perp ⅋ B^\perp \\
(A ⅋ B)^\perp & \equiv & A^\perp \otimes B^\perp \\
(A \oplus B)^\perp & \equiv & A^\perp \& B^\perp \\
(A \& B)^\perp & \equiv & A^\perp \oplus B^\perp \\
(!A)^\perp & \equiv & ?A^\perp \\
(?A)^\perp & \equiv & !A^\perp \\
I^\perp & \equiv & \perp \\
\mathbf{f}^\perp & \equiv & \mathbf{t}
\end{array}$$

Clearly **CLL** is a *symmetric* logic, in that a sequent $\Gamma \vdash \Delta$ is equivalent to the sequent $\vdash \Gamma^\perp, \Delta$. We can then give a succint presentation of **CLL** as a one-sided sequent calculus in Figure 5.2.

Presenting these rules in a graphical way leads to a formulation known as *Proof Nets* [31]. It seems that proof nets are the natural deduction for **CLL**; Abramsky [1] gives a term assignment for proof nets and gives an execution technique inspired by Berry and Boudol's (highly parallel) Chemical Abstract Machine [17]. It remains future work to devise a more traditional [63] natural deduction formulation of **CLL**.

Categorically, most models proposed for **CLL** are extensions of Seely's model for **ILL** to *-autonomous categories [68, 9]. It would seem likely that the problems identified with Seely's model in Chapter 4 would apply to these models. It should be the case that extending a Linear category with a dualizing object gives a model of **CLL**, although this has not been checked in detail. It would be interesting to see if an analysis such as that used in Chapter 4 would give rise to a similar categorical model. A particularly interesting class of models is given by various categories of *games*. It would appear that there are close connections between **CLL** and game theory [2]. Indeed this connection has been used (along with the Girard translation) to give fully abstract semantics for sequential computation [3, 44]. Clearly this is an extremely exciting area for future work.

$$\frac{}{A \vdash A}\ \mathit{Identity} \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \mathit{Cut}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ \mathit{Exchange}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ \mathit{Exchange}_{\mathcal{R}}$$

$$\frac{}{\Gamma \vdash \mathbf{t}, \Delta}\ (\mathbf{t}_{\mathcal{R}}) \qquad \frac{}{\Gamma, \mathbf{f} \vdash \Delta}\ (\mathbf{f}_{\mathcal{L}}) \qquad \frac{}{\bot \vdash}\ (\bot_{\mathcal{L}}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ (\bot_{\mathcal{R}})$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta}\ (I_{\mathcal{L}}) \qquad \frac{}{\vdash I}\ (I_{\mathcal{R}}) \qquad \frac{}{\Gamma \vdash \top, \Delta}\ (\top_{\mathcal{R}})$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, A \multimap B, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ (\&_{\mathcal{L}-1}) \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ (\&_{\mathcal{L}-2}) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta}\ (\&_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\ (\oplus_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ (\oplus_{\mathcal{R}-1}) \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ (\oplus_{\mathcal{R}-2})$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \parr B \vdash \Delta_1, \Delta_2}\ (\parr_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ (\parr_{\mathcal{R}})$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta}\ ({}^{\perp}{}_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}\ ({}^{\perp}{}_{\mathcal{R}})$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathit{Weakening}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\ \mathit{Weakening}_{\mathcal{R}}$$

$$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathit{Contraction}_{\mathcal{L}} \qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\ \mathit{Contraction}_{\mathcal{R}}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathit{Dereliction}_{\mathcal{L}} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta}\ \mathit{Dereliction}_{\mathcal{R}}$$

$$\frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}\ \mathit{Promotion}_{\mathcal{L}} \qquad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ \mathit{Promotion}_{\mathcal{R}}$$

Figure 5.1: Two-Sided Sequent Calculus Formulation of **CLL**

$$\frac{}{\vdash A^{\perp}, A}\ \textit{Identity} \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\ \textit{Exchange}$$

$$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \textit{Cut} \qquad \frac{}{\vdash I}\ I \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_2$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \textit{Weakening} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \textit{Contraction}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \textit{Dereliction} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ \textit{Promotion}$$

Figure 5.2: One-Sided Sequent Calculus Formulation of **CLL**

4 Further Work

There are many areas for further work arising from this thesis. In this section we shall consider some of them.

4.1 Further Categorical Analysis

Although the categorical analysis of Chapter 4 is thorough, there are still a number of interesting leads to follow. In particular further work is needed on finding a suitable subcategory of coalgebras which is cartesian closed and has coproducts.

4.2 Applications to Functional Programming

There are essentially two approaches to using linear logic for functional programming. The first is to consider a functional language which is based on the linear term calculus in the same way that existing functional languages are based on the λ-calculus. Some proposals have been made [57, 20] although it is probably fair to say that more theoretical work is needed.

Lafont [46] showed how an abstract machine based on linear logic needed no garbage collector. However, it should be noted that this depends on an implementation based on *copying*, whereas most implementations use *sharing*. It is not clear if linear logic can be used to give a logical foundation for sharing techniques. The second use of linear logic is to translate functional languages into the linear term calculus and use some of the fine grain information to suggest possible optimizations. This translation is essentially that presented in Chapter 3. However, that translation involves a large number of exponentials. It is then desirable to *minimize* the number of exponentials used in the translation. Some preliminary results exist [18, 24] on translating functional languages using a minimal number of exponentials, but it remains to be seen whether this technique will yield more efficient functional language implementations.

There is still work remaining concerning the correct notion of *canonical form* for the linear term calculus. When implementing a functional language, we introduce an *operational semantics*, which consists of a relation, written '$\Downarrow$', between terms and canonical forms. Normally these operational semantics correspond to the β-reduction rules which are applied using a *particular strategy* (e.g. call-by-value or call-by-name) until a particular form of term is reached. It is not instantly clear whether this can be quite so simply applied for the linear term calculus. Certainly the rules employed by others [1, 57, 55] do not seem to have this property. For example, consider the rule for the discard constructor [1]

$$\frac{N \Downarrow c}{\mathsf{discard}\ M\ \mathsf{in}\ N \Downarrow c}\ \textit{Discard.}$$

There is an additional problem with the explicit discard and copy constructors, as we noted in the proof of strong normalization; once one of these constructors is on the 'outside' of a term, it never disappears. However, we certainly do not want these term constructors to get in the way of evaluation. Consider the reduction sequence

$$MN \leadsto_\beta^+ (\mathsf{discard}\ \vec{P}\ \mathsf{in}\ \lambda x\colon A.Q)R.$$

At this stage the term is in β-normal form, but we clearly would like to apply a commuting conversion to allow the subsequent reduction sequence

$$(\mathsf{discard}\ \vec{P}\ \mathsf{in}\ \lambda x\colon A.Q)R \leadsto_c \mathsf{discard}\ \vec{P}\ \mathsf{in}\ (\lambda x\colon A.Q)R \leadsto_\beta \mathsf{discard}\ \vec{P}\ \mathsf{in}\ Q[x := R] \leadsto_\beta^* S.$$

This suggests that actually we should implement some of the commuting conversions within the operational semantics. Exactly how this would take shape remains future work. It is also interesting to note that the commuting conversions for the copy constructor are reminiscent of some of the manipulations performed during an optimal reduction strategy [5]. Again this is another area for future work.

4.3 Quantifiers

We have avoided completely the question of quantifiers for **ILL**. Proof theoretically these correspond to the following rules (with the traditional restriction on the $\forall_\mathcal{R}$ and $\exists_\mathcal{L}$ rules).

$$\frac{\Gamma, A[\alpha/\beta] \vdash B}{\Gamma, \forall\beta.A \vdash B}\ (\forall_\mathcal{L}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall\alpha.A}\ (\forall_\mathcal{R})$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \exists\alpha.A \vdash B}\ (\exists_\mathcal{L}) \qquad \frac{\Gamma \vdash A[\alpha/\beta]}{\Gamma \vdash \exists\beta.A}\ (\exists_\mathcal{R})$$

At the level of terms, such a system can be adapted to give a *Linear* System F [1]. A thorough investigation of this refinement of System F remains future work. In particular it would be interesting to see if the refinement of the connectives offers useful new representations of inductive datatypes [34, §11.5].

Seely [69] proposed modelling the quantifiers by moving to an indexed category. Ambler [4] has also considered various categorical models for (exponential free) linear logic with first order quantifiers.

4.4 Intuitionistic Modal Logics

The techniques developed in this thesis apply to a whole range of logics which have modalities. As we mentioned before the rules of *Dereliction* and *Promotion* correspond to the modality rules for **S4**-like modalities. A study of the necessity modality for intuitionistic **S4** has been performed by the author and de Paiva [19]. Studies of other modal logics remains future work.[3]

From a slightly different perspective, the author with de Paiva and Benton [13] have shown how the computational λ-calculus of Moggi [58] corresponds via the Curry Howard correspondence to a logic with a slightly unusual (at least from a traditional modal viewpoint) possibility modality. In particular this work has highlighted the logical status of the axioms of the computational λ-calculus. From a categorical perspective, Benton [11] has been investigating relationships between the computational λ-calculus and the linear term calculus.

[3] It should be noted that the field of *intuitionistic* modal logics is relatively unexplored, although Simpson [70] has carried out a recent study.

Bibliography

[1] S. Abramsky. Computational interpretations of linear logic. *Theoretical Computer Science*, 111(1–2):3–57, 1993. Previously Available as Department of Computing, Imperial College Technical Report 90/20, 1990.

[2] S. Abramsky and R. Jagadeesan. Games and full completeness for multiplicative linear logic. Technical Report 92/24, Department of Computing, Imperial College, London, September 1992.

[3] S. Abramsky, R. Jagadeesan, and P. Malacaria. Games and full abstraction for PCF. Unpublished Manuscript, July 1993.

[4] S.J. Ambler. *First Order Linear Logic in Symmetric Monoidal Closed Categories.* PhD thesis, Laboratory for Foundations of Computer Science, Department of Computer Science, University of Edinburgh, 1992.

[5] A. Asperti. Linear logic, comonads and optimal reductions. Unpublished Manuscript, April 1993.

[6] A. Asperti and G. Longo. *Categories, Types and Structures (An Introduction to Category Theory for the Working Computer Scientist).* MIT Press, Cambridge, Mass., 1991.

[7] A. Avron. The semantics and proof theory of linear logic. *Theoretical Computer Science*, 57:161–184, 1988.

[8] H.P. Barendregt. *The Lambda Calculus: Its Syntax and Semantics*, volume 103 of *Studies in logic and the foundations of mathematics.* North-Holland, revised edition, 1984.

[9] M. Barr. **-autonomous categories*, volume 752 of *Lecture Notes in Mathematics.* Springer-Verlag, 1979.

[10] M. Barr. *-autonomous categories and linear logic. *Mathematical Structures in Computer Science*, 1:159–178, 1991.

[11] P.N. Benton. A mixed linear and non-linear λ-calculus. Unpublished Manuscript, 1993.

[12] P.N. Benton. Strong normalization for the linear term calculus. Technical Report 305, Computer Laboratory, University of Cambridge, July 1993. To appear in *Journal of Functional Programming.*

[13] P.N. Benton, G.M. Bierman, and V.C.V. de Paiva. Computational types from a logical perspective I. Unpublished Manuscript, August 1993.

[14] P.N. Benton, G.M. Bierman, V.C.V. de Paiva, and J.M.E. Hyland. Linear λ-calculus and categorical models revisited. In E. Börger et al., editors, *Proceedings of The Sixth Workshop on Computer Science Logic*, volume 702 of *Lecture Notes in Computer Science*, pages 61–84, 1992.

[15] P.N. Benton, G.M. Bierman, V.C.V. de Paiva, and J.M.E. Hyland. Term assignment for intuitionistic linear logic. Technical Report 262, Computer Laboratory, University of Cambridge, August 1992.

[16] P.N. Benton, G.M. Bierman, V.C.V. de Paiva, and J.M.E. Hyland. A term calculus for intuitionistic linear logic. In M. Bezem and J.F. Groote, editors, *Proceedings of The International Conference on Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 75–90, 1993.

[17] G. Berry and G. Boudol. The chemical abstract machine. In *Proceedings of Symposium on Principles of Programming Languages*, pages 81–94, 1990.

[18] G.M. Bierman. Type systems, linearity and functional languages. In *Proceedings of CLICS Workshop*, pages 71–92, March 1992. Available as Aarhus University Technical Report DAIMI PB 397-I.

[19] G.M. Bierman and V.C.V. de Paiva. Intuitionistic necessity revisited. In *Proceedings of Logic at Work Conference. Amsterdam, Holland*, December 1992. To appear in forthcoming book: *Arrow Logics and Multi-modal Logics.*

[20] J. Chirimar, C.A. Gunter, and J.G. Riecke. Proving memory management invariants for a language based on linear logic. Technical Report MS-CIS-91-98, Department of Computer and Information Science, School of Engineering and Applied Science, University of Pennsylvania, December 1991.

[21] G. Cousineau, P.-L. Curien, and M. Mauny. The categorical abstract machine. In *Proceedings of Conference on Functional Programming Languages and Computer Architecture*, pages 50–64. ACM, 1985.

[22] R.L. Crole. Categories, equational logic and typed lambda calculi. Notes for a Graduate Lecture Course, September 1990.

[23] H.B. Curry and R. Feys. *Combinatory Logic*, volume 1. North Holland, second edition, 1968.

[24] V. Danos, J.-B. Joinet, and H. Schellinx. The structure of exponentials: uncovering the dynamics of linear logic proofs. Technical Report ML–93–10, Institute for Logic, Language and Computation, University of Amsterdam, May 1993.

[25] V.C.V. de Paiva and E. Ritter. Syntactic multicategories. Unpublished Manuscript, 1993.

[26] A.G. Dragalin. *Mathematical Intuitionism—Introduction to Proof Theory.* American Mathematical Society, 1988.

[27] S. Eilenberg and G.M. Kelly. Closed categories. In *Proceedings of Conference on Categorical Algebra, La Jolla*, 1966.

[28] A. Filinski. Linear continuations. In *Proceedings of Symposium on Principles of Programming Languages*, pages 27–38, January 1992.

[29] J. Gallier. Constructive logics part I: A tutorial on proof systems and typed λ-calculi. *Theoretical Computer Science*, 110(2):249–339, March 1993.

[30] J. Gallier. On the correspondence between proofs and λ-terms. Unpublished Manuscript, January 1993.

[31] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–101, 1987.

[32] J.-Y. Girard. The system F of variable types, fifteen years later. In G. Huet, editor, *Logical Foundations of Functional Programming*, chapter 6. Addison-Wesley, 1990.

[33] J.-Y. Girard and Y. Lafont. Linear logic and lazy computation. In *Proceedings of TAPSOFT 87*, volume 250 of *Lecture Notes in Computer Science*, pages 52–66, 1987. Previously Available as INRIA Report 588, 1986.

[34] J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*, volume 7 of *Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, 1989.

[35] J.-Y. Girard, A. Scedrov, and P.J. Scott. Bounded linear logic: a modular approach to polynomial time computability. *Theoretical Computer Science*, 97:1–66, 1992.

[36] R.P. Goré. *Cut-free Sequent and Tableau Systems for Propositional Normal Modal Logics.* PhD thesis, Computer Laboratory, University of Cambridge, 1992. Available as Computer Laboratory Technical Report 257.

[37] V. Harnik and M. Makkai. Lambek's categorical proof theory and Läuchli's abstract realizability. *The Journal of Symbolic Logic*, 57(1):200–230, March 1992.

[38] W.H. Hesselink. Axioms and models of linear logic. *Formal Aspects of Computing*, 2:139–166, 1990.

[39] J.R Hindley and J.P. Seldin. *Introduction to Combinators and λ-Calculus*, volume 1 of *London Mathematical Society Student Texts.* Cambridge University Press, 1986.

[40] W. Hodges. Elementary predicate logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic. Volume 1*, chapter 1, pages 1–131. D. Reidel, 1983.

[41] W.A. Howard. The formulae-as-types notion of construction. In J.R. Hindley and J.P. Seldin, editors, *To H.B. Curry: Essays on combinatory logic, lambda calculus and formalism.* Academic Press, 1980.

[42] P. Hudak, S. Peyton Jones, and P. Wadler (Eds). Haskell: A non-strict, purely functional language. *ACM SIGPLAN Notices*, 27(5), May 1992.

[43] J.M.E. Hyland and V.C.V. de Paiva. Full intuitionistic linear logic. Unpublished Manuscript. To appear in *Journal of Symbolic Logic*, 1993.

[44] J.M.E. Hyland and C.-H.L. Ong. Dialogue games and innocent strategies: An approach to (intensional) full abstraction for PCF. Unpublished Manuscript, July 1993.

[45] B. Jacobs. Conventional and linear types in a logic of coalgebras. Unpublished Manuscript, April 1993.

[46] Y. Lafont. The linear abstract machine. *Theoretical Computer Science*, 59:157–180, 1988. Corrections *ibid.* 62:327–328, 1988.

[47] Y. Lafont. *Logiques, Catégories et Machines.* PhD thesis, Université de Paris 7, 1988.

[48] J. Lambek. Deductive systems and categories I. *Mathematical Systems Theory*, 2:287–318, 1968.

[49] J. Lambek. Deductive systems and categories II. In P. Hilton, editor, *Category Theory, Homology Theory and their applications*, volume 86 of *Lecture Notes in Mathematics*, pages 76–122, 1969.

[50] J. Lambek. Deductive systems and categories III. In F.W. Lawvere, editor, *Toposes, algebraic geometry and logic*, volume 274 of *Lecture Notes in Mathematics*, pages 57–82, 1972.

[51] J. Lambek. Multicategories revisited. In *Conference on Categories in Computer Science and Logic*, volume 92 of *AMS Contemporary Mathematics*, pages 217–239, June 1989.

[52] J. Lambek and P.J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Cambridge studies in advanced mathematics.* Cambridge University Press, 1987.

[53] S. Mac Lane. *Categories for the Working Mathematican*, volume 5 of *Graduate Texts in Mathematics.* Springer Verlag, 1971.

[54] F.W. Lawvere. Adjointness in foundations. *Dialetica*, 23(3–4):281–296, 1969.

[55] P. Lincoln and J. Mitchell. Operational aspects of linear lambda calculus. In *Proceedings of Symposium on Logic in Computer Science*, pages 235–246, June 1992.

[56] F.E.J. Linton. Coequalizers in categories of coalgebras. In B. Eckman, editor, *Seminar on Triples and Categorical Homology Theory*, volume 80 of *Lecture Notes in Mathematics*, pages 75–90, 1969.

[57] I. Mackie. Lilac: A functional programming language based on linear logic. Master's thesis, Department of Computing, Imperial College, London, September 1991.

[58] E. Moggi. Computational lambda-calculus and monads. In *Proceedings of Symposium on Logic in Computer Science*, pages 14–23, June 1989.

[59] P.W. O'Hearn. Linear logic and interference control (preliminary report). In *Proceedings of Conference on Category Theory and Computer Science*, volume 530 of *Lecture Notes in Computer Science*, pages 74–93, September 1991.

[60] A.M. Pitts. Notes on categorical logic. Unpublished Manuscript, 1991.

[61] D. Prawitz. *Natural Deduction*, volume 3 of *Stockholm Studies in Philosophy*. Almqvist and Wiksell, 1965.

[62] D. Prawitz. Ideas and results in proof theory. In J.E. Fenstad, editor, *Proceedings of Second Scandinavian Logic Symposium*, pages 235–307, 1971.

[63] D. Prawitz and P.-E. Malmnäs. A survey of some connections between classical, intuitionistic and minimal logic. In A. Schmidt, K. Schütte, and H.J. Thiele, editors, *Contributions to Mathematical Logic*, pages 215–229. North-Holland, 1968.

[64] D. Roorda. *Resource Logics: Proof-Theoretical Investigations.* PhD thesis, University of Amsterdam, 1991.

[65] B. Russell and A.N. Whitehead. *Principia Mathematica.* Cambridge University Press, 1910.

[66] P. Schroeder-Heister. A natural extension of natural deduction. *The Journal of Symbolic Logic*, 49(4):1284–1300, December 1984.

[67] H. Schwichtenberg. Proof theory: Some applications of cut-elimination. In J. Barwise, editor, *Handbook of Mathematical Logic*, chapter D.2, pages 867–896. North Holland, 1977.

[68] R.A.G. Seely. Categorical semantics for higher order polymorphic lambda calculus. *The Journal of Symbolic Logic*, 52(4):969–989, 1987.

[69] R.A.G. Seely. Linear logic, *-autonomous categories and cofree algebras. In *Conference on Categories in Computer Science and Logic*, volume 92 of *AMS Contemporary Mathematics*, pages 371–382, June 1989.

[70] A. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logics.* PhD thesis, Laboratory for Foundations of Computer Science, Department of Computer Science, University of Edinburgh, 1993. To appear.

[71] W.R. Stoye. *The Implementation of Functional Languages using Custom Hardware.* PhD thesis, Computer Laboratory, University of Cambridge, 1985. Available as Computer Laboratory Technical Report 81.

[72] A. Suárez. Compiling ML into CAM. In G. Huet, editor, *Logical Foundations of Functional Programming*, chapter 4. Addison-Wesley, 1990.

[73] M.E. Szabo, editor. *The Collected Papers of Gerhard Gentzen.* North-Holland, 1969.

[74] W.W. Tait. Intensional interpretation of functionals of finite type. *The Journal of Symbolic Logic*, 32:198–212, 1967.

[75] A.S. Troelstra. *Lectures on Linear Logic*, volume 29 of *Lecture Notes.* CSLI, 1992.

[76] A.S. Troelstra. Natural deduction for intuitionistic linear logic. Technical Report ML–93–09, Institute for Logic, Language and Computation, University of Amsterdam, May 1993.

[77] D.A. Turner. A new implementation technique for applicative languages. *Software—Practice and Experience*, 9:31–49, 1979.

[78] D.A. Turner. Miranda: a non-strict functional language with polymorphic types. In *Proceedings of Conference on Functional Programming Languages and Computer Architecture*, volume 201 of *Lecture Notes in Computer Science*, pages 1–16, 1985.

[79] P. Wadler. There's no substitute for linear logic. Paper presented at Workshop on Programming Language Semantics, Oxford, April 1992.