Ternary and Three-point Univariate Subdivision Schemes

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Technical Report Number 520
University of Cambridge
Computer Laboratory‡

September 6, 2001

Abstract

The generating function formalism is used to analyze the continuity properties of univariate ternary subdivision schemes. These are compared with their binary counterparts.

1 Introduction

Most work in the area of subdivision schemes has considered binary schemes with an even number of control points. Following a similar argument to that used in [2], we decided to investigate schemes with an odd number of control points, specifically 3-point schemes. This led to a more general investigation of ternary subdivision schemes.

For symmetry reasons, it is obvious that an interpolating binary subdivision scheme which utilizes the closest k points, for k odd, reduces to a scheme which utilizes just the closest $k-1$ points, $k-1$ even. There is thus no 3-point interpolating binary subdivision scheme. Ternary subdivision, on the other hand, does allow for an interpolating 3-point subdivision scheme. A family of such schemes has been shown to exist and have C^1 continuity, as demonstrated later in this report. Further investigation led to discovery of a family of interpolating 4-point ternary subdivision schemes which have C^2 continuity [5].

Investigation of approximating 3-point schemes has led to two interesting subdivision schemes. An approximating 3-point ternary scheme has been found and shown to have C^2 continuity. An approximating 3-point binary scheme, which uses corner-cutting similar in spirit to the 2-point scheme $\frac{1}{4}[1,3,3,1]$, can be derived and shown to have C^3 continuity. Both schemes are presented in full later in this report.

We have investigated these schemes using the generating function formalism, which lends itself well to deriving sufficient conditions for subdivision schemes

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Interpolating		
Scheme	Highest continuity	Mask
2-point	C^0	$\frac{1}{3}[1,2,3,2,1]$
3-point	C^1	$[a,0,b,1-a-b,1,1-a-b,b,0,a]$
4-point	C^2	$[a_3,a_0,0,a_2,a_1,1,a_1,a_2,0,a_0,a_3]$
Approximating		
3-point	C^2	$\frac{1}{27}[1,4,10,16,19,16,10,4,1]$

where

$$
\begin{aligned}
a &= b-\frac{3}{9}\\
a_0 &= -\frac{1}{18}-\frac{1}{6}\mu\\
a_1 &= \frac{13}{18}+\frac{1}{2}\mu\\
a_2 &= \frac{7}{18}-\frac{1}{2}\mu\\
a_3 &= -\frac{1}{18}+\frac{1}{6}\mu
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{2}{9} < & \; b \; < \frac{3}{9}\\
\frac{1}{9} < & \; \mu \; < \frac{1}{15}
\end{aligned}
$$

Figure 1: Table showing results for ternary schemes.

to be C^k. For binary schemes the subdivision step can be compactly written in a single equation

$$p_j^{i+1} = \sum_{k\in\mathbb{Z}} \alpha_{(2k-j)} p_k^i, \tag{1}$$

and similarly for ternary schemes

$$p_j^{i+1} = \sum_{k\in\mathbb{Z}} \alpha_{(3k-j)} p_k^i, \tag{2}$$

where $\alpha = (\alpha_j)$ is the mask of the scheme and p^i are the set of points after the i^{th} subdivision step. The principal results for the ternary schemes are tabulated in Figure 1 and compared with those for binary schemes tabulated in Figure 2.

The results for the 2-point interpolating schemes are trivial. [5] shows the derivation of the ternary 4-point interpolating scheme. [4] derives the results for the binary 4-point interpolating scheme and [7] derives the results for the binary 6-point interpolating scheme. Chaikin first proposed the binary 2-point approximating scheme in [1], which was shown to produce the quadratic b-spline

Interpolating		
Scheme	Highest continuity	Mask
2-point	C^0	$\frac{1}{2}[1,2,1]$
4-point	C^1	$\frac{1}{16}[-1,0,9,16,9,-1]$
6-point	C^2	$[\theta,0,-3\theta-\frac{1}{16},0,2\theta+\frac{9}{16},1,2\theta+\frac{9}{16},0,-3\theta-\frac{1}{16},0,\theta]$
Approximating		
2-point	C^1	$\frac{1}{4}[1,3,3,1]$
3-point	C^3	$\frac{1}{16}[1,5,10,10,5,1]$

where $0<\theta<0.02$.

Figure 2: Table showing results for binary schemes.

at the limit [6] and it can be shown that the ternary 3-point approximating scheme produces the cubic B-spline at the limit and that the binary 3-point approximating scheme produces the quartic B-spline.

In the following we will describe the generating function formalism and how it is used to derive continuity.

2 Generating function formalism

2.1 Binary Schemes

From the method of Dyn [3], after some computation we, see that the subdivision step for binary schemes can be expressed in the generating function formalism as a simple multiplication of the corresponding symbols:

$$P^{i+1}(z)=\alpha(z)P^i(z^2), \tag{3}$$

where

$$P^i(z)=\sum_j p^i_j z^j, \alpha(z)=\sum_j \alpha_j z^j. \tag{4}$$

2.1.1 Sufficient conditions for C^k

Now we will state sufficient conditions for a binary scheme to be C^k. The proof is given in [3].

For any given binary subdivision scheme, S, with a mask α satisfying (5), we can prove $S^\infty P^0 \in C^k$ by first deriving the mask of $\frac{1}{2}S_{k+1}$ and then computing $||(\frac{1}{2}S_{k+1})^i||_\infty$ for $i=1,2,3,\dots L$, where L is the first integer for which $||(\frac{1}{2}S_{k+1})^L||_\infty<1$. If such an L exists and the mask of S_l satisfies (5) $\forall l\le k$ then $S^\infty P^0\in C^k$.

$$\sum_{j\in\mathbb{Z}}\alpha_{2j}=1, \sum_{j\in\mathbb{Z}}\alpha_{2j+1}=1. \tag{5}$$

$$\left|\left|\frac{1}{2}S_{k+1}\right|\right|_\infty=\frac{1}{2}\max\left(\sum_{j\in\mathbb{Z}}\left|\alpha^{(k+1)}_{2j}\right|,\sum_{j\in\mathbb{Z}}\left|\alpha^{(k+1)}_{2j+1}\right|\right) \tag{6}$$

where

$$
\begin{aligned}
& 2z[\alpha^{(k)}(z)] && = [\alpha^{(k+1)}(z)](1+z) && (7)\\
\Rightarrow \quad & 2\alpha_i^{(k)} && = \alpha_{i-1}^{(k+1)} + \alpha_i^{(k+1)} && (8)
\end{aligned}
$$

2.2 Ternary Schemes

Again following the method of Dyn [3], after some computation, we see that the subdivision step for ternary schemes can be expressed in the generating function formalism as a simple multiplication of the corresponding symbols:

$$P^{i+1}(z) = \alpha(z)P^i(z^3), \tag{9}$$

where

$$P^i(z) = \sum_j p_j^i z^j, \alpha(z) = \sum_j \alpha_j z^j. \tag{10}$$

2.2.1 Sufficient conditions for C^k

Now we will state sufficient conditions for a ternary scheme to be C^k. The proof is given in [5].

For any given ternary subdivision scheme, S, with a mask α satisfying (11), we can prove $S^\infty P^0 \in C^k$ by first deriving the mask of $\frac{1}{3}S_{k+1}$ and then computing $||(\frac{1}{3}S_{k+1})^i||_\infty$ for $i = 1, 2, 3, \ldots L$, where L is the first integer for which $||(\frac{1}{3}S_{k+1})^L||_\infty < 1$. If such an L exists and the mask of S_l satisfies (11) $\forall l \le k$ then $S^\infty P^0 \in C^k$.

$$\sum_{j\in\mathbb{Z}} \alpha_{3j} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+1} = 1, \sum_{j\in\mathbb{Z}} \alpha_{3j+2} = 1. \tag{11}$$

$$\left|\left|\frac{1}{3}S_{k+1}\right|\right|_\infty = \frac{1}{3}\max\left(\sum_{j\in\mathbb{Z}}\left|\alpha_{3j}^{(k+1)}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha_{3j+1}^{(k+1)}\right|, \sum_{j\in\mathbb{Z}}\left|\alpha_{3j+2}^{(k+1)}\right|\right) \tag{12}$$

where

$$
\begin{aligned}
& 3z^2[\alpha^{(k)}(z)] && = [\alpha^{(k+1)}(z)](1+z+z^2) && (13)\\
\Rightarrow \quad & 3\alpha_i^{(k)} && = \alpha_{i-2}^{(k+1)} + \alpha_{i-1}^{(k+1)} + \alpha_i^{(k+1)} && (14)
\end{aligned}
$$

3 Continuity of interpolating 3-point ternary subdivision

For this scheme we have

$$
\begin{aligned}
\alpha &= [\quad \ldots, 0, 0, a, 0, b, 1-a-b, 1, 1-a-b, b, 0, a, 0, 0, \ldots] && (15)\\
\alpha^{(1)} &= 3[\quad \ldots, 0, 0, a, -a, b, 1-2b, b, -a, a, 0, 0, \ldots] && (16)
\end{aligned}
$$

It is easy to verify that α satisfies (11).

If

$$\left|\left|\frac{1}{3}S_1\right|\right|_\infty = \max\left(|1-2b|+2|a|, |a|+|b|, |a|+|b|\right) < 1 \tag{17}$$

then this scheme has C^0 continuity.

Now for C^1 continuity we first need $\alpha^{(1)}$ to satisfy (11). This implies

$$a = b - \frac{1}{3}. \tag{18}$$

and so we have

$$\alpha^{(1)} = 3[\quad \ldots, 0, 0, b-\frac{1}{3}, \frac{1}{3}-b, b, 1-2b, b, \frac{1}{3}-b, b-\frac{1}{3}, 0, 0, \ldots] \tag{19}$$

$$\alpha^{(2)} = 9[\quad \ldots, 0, 0, b-\frac{1}{3}, \frac{2}{3}-2b, 2b-\frac{1}{3}, \frac{2}{3}-2b, b-\frac{1}{3}, 0, 0, \ldots] \tag{20}$$

If

$$\left|\left|\frac{1}{3}S_2\right|\right|_\infty = \max\left(9\left|b-\frac{1}{3}\right|, 9\left|b-\frac{1}{3}\right|, 3\left|2b-\frac{1}{3}\right|\right) < 1 \tag{21}$$

then we have C^1 continuity.

$$\frac{2}{9} < b < \frac{3}{9}, a = b - \frac{3}{9} \tag{22}$$

$$\tag{23}$$

satisfies (17) and (21).

For C^2 continuity we would require $\alpha^{(2)}$ to satisfy (11). This implies $b = \frac{2}{9}$, but with this value $\left|\left|\frac{1}{3}S_2\right|\right|_\infty = 1$. For $b = \frac{2}{9}$

$$\alpha^{(2)} = [\quad \ldots, 0, 0, -1, 2, 1, 2, -1, 0, 0, \ldots] \tag{24}$$

$$(\alpha^{(2)})^2 = [\quad \ldots, 0, 0, 1, -4, 2, 0, 11, 0, 2, -4, 1, 0, 0, \ldots] \tag{25}$$

Hence $\left|\left|\left(\frac{1}{3}S_2\right)^2\right|\right|_\infty = \frac{19}{9}$. Thus it looks as if there is no C^2 3-point interpolating ternary subdivision scheme.

Hence a C^1 ternary 3-point interpolating subdivision scheme can be defined by (15), where $\frac{2}{9} < b < \frac{3}{9}$ and $a = b - \frac{1}{3}$.

4 Continuity of the approximating 3-point ternary scheme

There are several ways to arrive at this scheme. One is through the generating function formalism as followed above, another method is to use the matrix formalism, and finally it can be arrived at from the cubic B-spline itself. Here we will just prove the continuity of the scheme using the generating function formalism.

For this scheme we have

$$\alpha = \tfrac{1}{27}[\quad \ldots, 0, 0, 1, 4, 10, 16, 19, 16, 10, 4, 1, 0, 0, \ldots] \tag{26}$$

$$\alpha^{(1)} = \tfrac{1}{9}[\quad \ldots, 0, 0, 1, 3, 6, 7, 6, 3, 1, 0, 0, \ldots] \tag{27}$$

It is easy to verify that α satisfies (11).

$$\left|\left|\frac{1}{3}S_1\right|\right|_\infty = \max\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) < 1 \tag{28}$$

Hence this scheme has C^0 continuity.

Now for C^1 continuity we first need $\alpha^{(1)}$ to satisfy (11), which it does. Now

$$\begin{aligned} \alpha^{(2)} &= \frac{1}{3}[\ldots, 0, 0, 1, 2, 3, 2, 1, 0, 0, \ldots] && (29)\\ \Rightarrow \quad \left|\left|\tfrac{1}{3}S_2\right|\right|_\infty &= \max\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) < 1 && (30) \end{aligned}$$

Hence this scheme has C^1 continuity.

Now for C^2 continuity we first need $\alpha^{(2)}$ to satisfy (11), which it does. Now

$$\begin{aligned} \alpha^{(3)} &= [\ldots, 0, 0, 1, 1, 1, 0, 0, \ldots] && (31)\\ \Rightarrow \quad \left|\left|\tfrac{1}{3}S_3\right|\right|_\infty &= \max\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) < 1 && (32) \end{aligned}$$

Hence this scheme has C^2 continuity.

Now for C^3 continuity we first need $\alpha^{(3)}$ to satisfy (11), which it does. But

$$\begin{aligned} \alpha^{(4)} &= 3[\ldots, 0, 0, 1, 0, 0, \ldots] && (33)\\ \Rightarrow \quad \left|\left|\tfrac{1}{3}S_3\right|\right|_\infty &= \max(1, 0, 0) \geq 1 && (34)\\ (\alpha^{(4)})^2 &= 9[\ldots, 0, 0, 1, 0, 0, \ldots] && (35)\\ \Rightarrow \quad \left|\left|\left(\tfrac{1}{3}S_3\right)^2\right|\right|_\infty &= \max(1, 0, 0) \geq 1 && (36) \end{aligned}$$

Hence this scheme does not have C^3 continuity.

5 Continuity of 3-point approximating binary subdivision

This scheme can be easily derived from the quartic B-spline. Here we take a different approach. We start from the general form of a binary 3-point subdivision scheme. We then apply continuity requirements in order, showing that the quartic B-spline scheme is the only scheme of this type which has C^3-continuity, but that there are an infinite range of schemes with lower continuity.

There is no point in having a 3-point interpolating binary scheme, as such a scheme would reduce to the 2-point scheme: $\frac{1}{2}[1, 2, 1]$. However, a 3-point approximating binary scheme may be possible. This would be a corner-cutting scheme similar to the 2-point scheme: $\frac{1}{4}[1, 3, 3, 1]$. Its mask is

$$\alpha = [a, b, 1-a-b, 1-a-b, b, a] \tag{37}$$

For C^0 continuity we require that the mask satisfy (5), which it does, and $\left|\left|\frac{1}{2}S_1\right|\right|_\infty < 1$.

$$\begin{aligned} \alpha^{(1)} &= 2[\ldots, 0, 0, a, b-a, 1-2b, b-a, a, 0, 0, \ldots] && (38)\\ \Rightarrow \quad \left|\left|\tfrac{1}{2}S_1\right|\right|_\infty &= \max\left(|1-2b| + 2|a|, 2|b-a|\right) < 1 && (39) \end{aligned}$$

For C^1 continuity we require that $\alpha^{(1)}$ satisfy (5), which implies that $b = a + \frac{1}{4}$, and also $\left|\left|\frac{1}{2}S_2\right|\right|_\infty < 1$.

$$\alpha^{(2)} = 4[\ldots, 0, 0, a, \frac{1}{4} - a, \frac{1}{4} - a, a, 0, 0, \ldots] \quad (40)$$

$$\Rightarrow \left|\left|\tfrac{1}{2}S_2\right|\right|_\infty = 2|a| + 2\left|\frac{1}{4} - a\right| < 1 \quad (41)$$

For C^2 continuity we require that $\alpha^{(2)}$ satisfy (5), which is true, and also $\left|\left|\frac{1}{2}S_3\right|\right|_\infty < 1$.

$$\alpha^{(3)} = 8[\ldots, 0, 0, a, \frac{1}{4} - 2a, a, 0, 0, \ldots] \quad (42)$$

$$\Rightarrow \left|\left|\tfrac{1}{2}S_3\right|\right|_\infty = \max\left(|8a|, |1 - 8a|\right) < 1 \quad (43)$$

which implies that $0 < a < \frac{1}{8}$.

For C^3 continuity we require that $\alpha^{(3)}$ satisfy (5), which implies that $a = \frac{1}{16}$, which incidentally meets the criterion in equations (39) and (41), and also $\left|\left|\frac{1}{2}S_4\right|\right|_\infty < 1$.

$$\alpha^{(4)} = [\ldots, 0, 0, 1, 1, 0, 0, \ldots] \quad (44)$$

$$\Rightarrow \left|\left|\tfrac{1}{2}S_4\right|\right|_\infty = \max\left(\frac{1}{2}, \frac{1}{2}\right) < 1 \quad (45)$$

To go to C^4 continuity we require that $\alpha^{(4)}$ satisfy (5), which it does, and also $\left|\left|\frac{1}{2}S_5\right|\right|_\infty < 1$, which it does not:

$$\alpha^{(5)} = 8[\ldots, 0, 0, 2, 0, 0, \ldots] \quad (46)$$

$$\Rightarrow \left|\left|\tfrac{1}{2}S_5\right|\right|_\infty = 1 \quad (47)$$

Thus the limit curve for the binary scheme with the mask $\alpha = \frac{1}{16}[1, 5, 10, 10, 5, 1]$ has C^3 continuity.

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