

Number 542



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Computer Laboratory

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September 2002

An updated, improved version of this report has been published in
Computer Aided Geometric Design
22(1):99-109, Jan 2004

[doi:10.1016/j.cagd.2003.08.001].

The classification scheme is slightly different in the CAGD version. Please refer to the CAGD version in any work which you produce.

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Series editor: Markus Kuhn

ISSN 1476-2986

A Generative Classification of Mesh Refinement Rules with Lattice Transformations

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September 2002

Abstract

We give a classification of the subdivision refinement rules using sequences of similar lattices. Our work expands and unifies recent results in the classification of primal triangular subdivision [1], and results on the refinement of quadrilateral lattices [14] [13]. In the examples we concentrate on the cases with low ratio of similarity and find new univariate and bivariate refinement rules with the lowest possible such ratio, showing that this very low ratio usually comes at the expense of symmetry.

Keywords: subdivision; lattices.

1 Introduction

Recursive subdivision has recently emerged as an efficient way for the construction of high quality surfaces. The basic idea is to start with a coarse polyhedral mesh and progressively refine it, by adding new vertices and joining them with edges and faces, until at the limit of that process we get a smooth surface.

Most of the subdivision schemes are based on a set of rules applying on regular meshes, and a generalization of these rules so that they cover singular vertices. Recall that a vertex is called regular if has valency 6,4,3 for triangular, quadrilateral and hexagonal meshes, respectively, and irregular or singular otherwise. These valencies come from the corresponding regular tessellations of the Euclidean plane into triangles, quadrilaterals and hexagons.

There are several logical steps in the definition of a subdivision scheme.

1. The description of the sequence of regular meshes generated by the subdivision process.
2. The choice of the extent of the non-zero coefficients in the stencils, that is, the selection of the set of existing vertices that will be used to calculate each new vertex in the next step.
3. The choice of the exact values of those coefficients.
4. The modifications of the rules for the irregular case.

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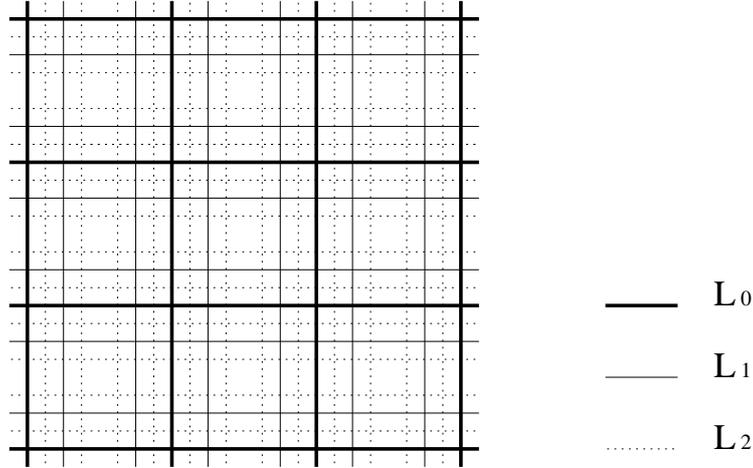


Figure 1: The biquadratic subdivision lattice generated by a non-primal scheme such as Doo-Sabin.

Here we deal just with the first step which usually characterizes a subdivision scheme as the other three steps are determined by taking into account other considerations, like the extent of the support, the fairness and the smoothness of the resulting surface.

Our Contribution: In this paper we unify and expand the classification of primal refinement rules for triangular meshes, Alexa [1], and the study of quadrilateral lattices in the context of numerical integration Sloan et. al [14] [13]. We find interesting connections and ramifications with algebra, geometry, combinatorics, even elementary number theory, gaining this way a further insight into the mathematics of the subdivision processes.

We do not study any particular subdivision scheme but we present a unified framework for the classification of subdivision schemes, univariate or bivariate, triangular, quadrangular or hexagonal, primal or dual, binary, ternary, or n -ary. Our approach is general enough to encompass all the known subdivision schemes. Moreover, it allows us to find some new refinement rules with interesting properties.

2 Sequences of lattices

Consider the vertices of a regular triangular or quadrilateral tessellation of the Euclidean plane. They form a lattice L_0 , that is, choosing arbitrarily one of them to be the origin O , and under the usual additive operation of \mathbf{R}^2 , which is:

$$P_1 + P_2 = P \quad \text{iff} \quad O\vec{P}_1 + O\vec{P}_2 = O\vec{P} \quad (1)$$

they form a subgroup G_0 of $(\mathbf{R}^2, +)$. In particular we have

$$G_0 \simeq \mathbf{Z} \times \mathbf{Z} \quad (2)$$

After the first iteration of the subdivision scheme the vertices of the new mesh form a new lattice L_1 with an underlying group G_1 . Continuing the subdivision process we get a sequence of similar lattices

$$L_0, L_1, L_2, \dots \quad (3)$$

see Fig. 1 for an example, and a sequence of underlying groups,

$$G_0, G_1, G_2, \dots \quad (4)$$

Alexa [1] studied the sequences of lattices (3) corresponding to regular triangular meshes. There, the implied conditions for a sequence of lattices to correspond to a valid subdivision scheme were

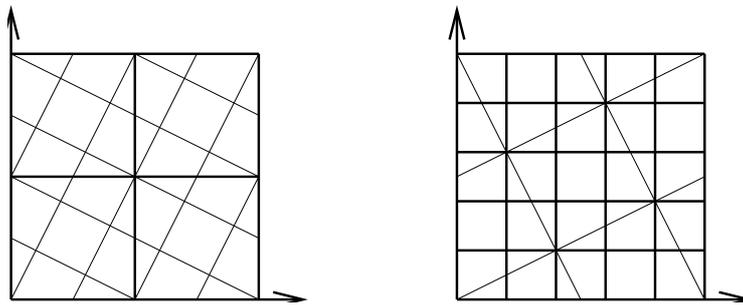


Figure 2: Left: the coordinate system is the coarse mesh. Right: the coordinate system is the fine mesh.

1. All the lattices are similar, in the sense of geometric Euclidean similarity.
2. The similarity between L_n and L_{n+1} , seen as a transformation of the Euclidean plane, is the same for $n = 0, 1, \dots$. The scale ratio of this similarity will be called the *arity* of the scheme.
3. The lattices, as point sets, form a nested sequence of proper inclusions.

$$L_0 \subset L_1 \subset L_2 \subset \dots \quad (5)$$

Under the first and second condition we see that the first two terms of the sequence (3) suffice to describe the subdivision process. After a normalisation¹ of the lattice L_1 , supposing that the two generating edges have length 1 and their angle is $\frac{\pi}{3}$ for triangular meshes $\frac{\pi}{2}$ for quad meshes, it suffices to consider the lattice L_0 . Taking into account the inclusion relation (5), we can easily prove that one edge of the tessellation corresponding to L_0 suffices to describe the subdivision process, see Alexa [1] for triangular meshes. Fig. 2 shows an example of normalising either the coarser or the finer mesh.

In this paper we keep the first condition of the similarity, so we deal only with the more interesting case of symmetric subdivision. We also keep the second condition, so that we do not have to analyse mixtures of different subdivision schemes. But we replace the third condition with another, weaker one, because the inclusion relation means that at each step we keep the vertices of the previous step, which means that we deal only with primal subdivision. The simplest weaker condition, given that we want to include non-primal schemes in our classification, is to consider the centres, as well as the vertices of L_1 , as admissible locations for the vertices of the coarser mesh L_0 .

In an alternative approach, noting that each vertex has a rotational symmetry of order equal to its valency, we could require that the vertices of the coarser mesh be fixed points of the symmetries of the finer one. In the case of the planar regular lattices we are mainly interested in, the points fixed by rotations would be vertices, face centres or edge midpoints. This alternative approach might be useful in more general situations, such as higher dimensions or semi-regular or irregular meshes, but we do not explore it in this paper.

3 The univariate case

Although the above exposition was in dimension 2, one of the advantages of our approach is that it is dimension free. In this section we will deal with the univariate case before moving to the

¹Note that we can have an equivalent exposition by normalising the lattice L_0 rather than the lattice L_1 . For example, Sloan et. al [14] deals with just two lattices rather than a sequence, for the purposes of numerical integration, and a lot of times the preferred normalisation is that of L_0 . The advantage from the normalisation of the finer mesh L_1 is that we have to handle vertices with integer coordinates only. This duality results from the invertibility of the transformation between L_0 and L_1 , and a detailed study of it can be found in Senechal [12]. It is particularly useful when we consider subdivision as an invertible rather than a refinement process.

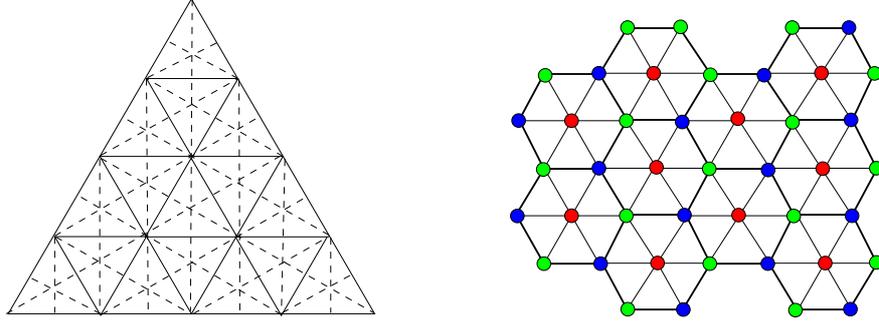


Figure 3: Left: The lattice L is shown with the dashed lines and the lattice L_1 with continuous lines. Right: The points of L colored in red, green, blue.

bivariate cases in the succeeding sections.

In the univariate case the normalised lattice L_1 is the set of integers \mathbf{Z} . The points fixed by a rotation are the vertices and the midedges which together form the lattice

$$L = \left\{ \frac{n}{2} \mid n \in \mathbf{Z} \right\} \quad (6)$$

which is similar to \mathbf{Z} . The sublattices L_0 of L can be described by one of their edges, i.e., the part of the real line between two adjacent vertices of L . We can distinguish between these sublattices by the length of their edge, measured with respect to L_1 , and by the kind of the ends of the edges, that is, if they are vertices or midedges of L_1 . This way we separate three distinct cases.

- If both ends are vertices of the finer mesh, then the length is an integer n , and the scheme we get is the n -ary refinement [4].
- If both ends are midedges, then again the length is an integer. If the length is equal to 2 we get schemes of which the classic Chaikin scheme is an example, while if the length is greater than 2 we get a combination of knot-insertion refinement and corner-cutting. The schemes of this category can be called dual because at each step the vertices of the initial polygonal curve correspond to midedges of the refined.
- If one of the ends is a vertex and the other is a midedge, the length has the form $\frac{2n+1}{2}$, $n = 1, 2, \dots$. This case is particularly interesting for $n = 1$ because then the arity is $\frac{3}{2}$, the lowest possible. The corresponding scheme enters new vertices at distance $\frac{1}{3}$ on the left and on the right of the odd vertices and then removes the original odd vertices. Such schemes involve a significant loss of symmetry, because some vertices at each level are handled in one way, other in another. We include these schemes here for completeness.

4 Triangular meshes

In the case of regular triangular meshes, the normalisation of L_1 means that its two generators are vectors of length 1 forming an angle of $\frac{\pi}{3}$. The fixed points of the rotational symmetries of L_1 are the vertices, fixed by a rotation of order 6, and the centrefaces, fixed by a rotation of order 3.

To classify all the possible schemes first we notice that these critical points also form a lattice L which is the lattice of L_1 after a refinement with the $\sqrt{3}$ operator [10]. See Fig. 3 (left).

We want to find all the possible lattices L_0 with vertices on the vertices of the lattice of critical points L . Although it is possible to work just algebraically, it will give a better insight if we first study the combinatorics of the lattice L . So, in a first step we separate the vertices of L into 3 distinct classes, or from another point of view we color them using three colors. The vertices of L corresponding to centrefaces of L_1 form a regular hexagonal mesh, which is the dual of L_1 . The

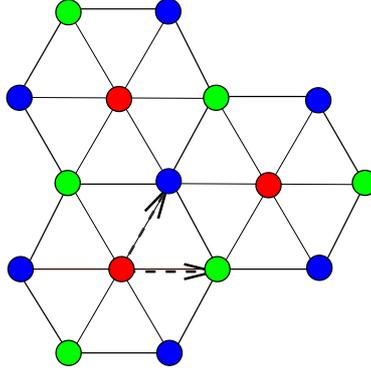


Figure 4: A coordinate system for the lattice L .

regular hexagonal mesh is a bipartite mesh, that is, its vertices can be separated into two classes, or be colored with two colors, such that only vertices of opposite colors are joined by an edge. This coloring is unique, up to a permutation of the colors and can be easily performed by arbitrarily choosing the color of a vertex, coloring its neighbors with the opposite color and continuing this way. The centres of this bipartite hexagonal mesh correspond to the vertices of L_1 and form the third class of points. Fig. 3 (right) shows the regular hexagonal mesh as a bipartite map, and the coloring of the vertices of L using the colors red, green, and blue.

We will use a coordinate system defined by the two generators of L , assuming that the points $(0,0)$, $(1,0)$, $(0,1)$ are red, green, blue, respectively, see Fig. 4. By induction we can easily prove that the coordinates of the red, green and blue points are (n, m) , $n, m \in \mathbf{Z}$, and

$$n - m \equiv 0 \pmod{3}, \quad n - m \equiv 1 \pmod{3}, \quad n - m \equiv 2 \pmod{3}. \quad (7)$$

corresponding.

As the red points correspond to vertices of L_1 , their difference from the green and blue, which correspond to centres, is obvious. The difference between the green and blue points is more subtle but also important for our purposes. Geometrically, looking at Fig. 3 (left), the triangles of L_1 with green centres point up (\triangle) while those with blue centres point down (∇). That distinction is possible because in a triangular grid the order of the rotational symmetry of the faces is 3, that is, half the order of the rotational symmetry of the vertices which is 6. This is not the case with quad grids where, as a result, the classification is easier, while in the hexagonal grids, through the duality with the triangular grids, we can distinguish two types of vertices.

We have

Proposition 4.1: Let $P_0P_1P_2$ be an equilateral triangle with vertices on the vertices of L . The points P_0, P_1, P_2 either all have the same color or all three have different colors.

Proof : Let $P_0 = (m_0, n_0)$, $P_1 = (m_1, n_1)$. The position of P_2 , assuming an anticlockwise orientation is

$$(n_0 + m_0 - m_1, -n_0 + n_1 + m_1) \quad (8)$$

The colors of the three points, as described in (7) are

$$\begin{aligned} (2n_0 - n_1 + m_0 - 2m_1) \pmod{3} \\ (n_0 - m_0) \pmod{3} \\ (n_1 - m_1) \pmod{3} \end{aligned} \quad (9)$$

corresponding. Their sum is $0 \pmod{3}$. Thus, to find all the possible combinations we have to solve the Diophantine equation

$$x + y + z \equiv 0 \pmod{3} \quad (10)$$

The solutions of this equation are given by x, y, z equal to each other, or mutually different to each other, mod 3. ■

The classification will be stated using the coordinate system of L_1 rather than L . This way our results will interpret more naturally the geometry of the subdivision process. In that case the coordinates of red, green and blue points are

$$(n, m) \quad (n + \frac{1}{3}, m + \frac{1}{3}) \quad (n + \frac{2}{3}, m + \frac{2}{3}) \quad n, m \in \mathbf{Z} \quad (11)$$

respectively, see Fig. 5.

So, according to the color of the vertices of the triangle $P_0P_1P_2$ of Proposition 4.1 we have the following types of subdivision schemes

1. All the vertices of the triangle $P_0P_1P_2$ are red, that is, vertices of L_1 .

In that case we have a primal scheme. Up to a translation of L_1 we may assume that the vertex P_0 is at the origin. Then a second vertex of the triangle, let say P_1 , suffices to determine the lattice L_0 , and determines it uniquely up to a rotation of order 6 with fixed point the origin. So, with the symmetry criteria and the coordinate system we use, the primal schemes are in 1-1 correspondence with the pairs of integers (n, m) after the identification of the pairs

$$\begin{aligned} &(n, m), (-m, n + m), (-n - m, n), \\ &(-n, -m), (m, -n - m), (n + m, -n) \end{aligned} \quad (12)$$

2. All the vertices of the triangle $P_0P_1P_2$ are centres of L_1 .

In that case we have a dual scheme. The points P_0, P_1, P_2 are either all green, or all blue. If they are green there is a translation of L_1 sending P_0 to $(\frac{1}{3}, \frac{1}{3})$, while if they are blue there is a rotation of L_1 followed by a translation of L_1 , sending P_0 to $(\frac{1}{3}, \frac{1}{3})$. Without loss of generality we suppose that P_0 is green. The lattice L_0 is determined by the position of the point P_1 which is also green, i.e., its coordinates are $(n + \frac{1}{3}, m + \frac{1}{3})$. The points P_1 are considered distinct, up to a rotation by $\frac{2\pi}{3}$ through $(\frac{1}{3}, \frac{1}{3})$.

3. The points P_0, P_1, P_2 all have different colors, i.e. two of them are centres of L_1 and one is vertex.

Notice that in this case we have a mixed primal-dual scheme, that is, in every step some vertices remain vertices while some others correspond to centres of the refined mesh. Assume that P_0 is the red vertex. Using a translation of L_1 we can send it to the origin and then the vertex P_1 which has coordinates of the form $(n + \frac{1}{3}, m + \frac{1}{3})$ or $(n + \frac{2}{3}, m + \frac{2}{3})$, defines L_0 up to rotation by $\frac{\pi}{3}$ through the origin. Equivalently, we may assume that the point P_1 is green and so its coordinates are $(n + \frac{1}{3}, m + \frac{1}{3})$, and its position define the lattice up to a rotation by $\frac{2\pi}{3}$ through the origin.

We expect that it is more difficult to extend these mixed primal-dual schemes to cover the irregular case, because in each step we have to determine which vertices are mapped to vertices and which are mapped to centres.

Notice that in both case 2 and in case 3 the triangle $P_0P_1P_2$ contains an edge with both ends centres. So, the distinction between these two cases requires something more subtle than the separation of the vertices of L into vertices and centres of L_1 . It is because we need to distinguish between cases 2 and 3 that we have to divide the centres of L in the two classes, green and blue. Notice, that the distinction between different kinds of centres introduces an asymmetry to any dual triangular scheme.

Looking for schemes with low arity we notice that the smallest distance greater than 1 on L is $\frac{2\sqrt{3}}{3}$, see Fig. 5 (left). It gives a mixed dual-primal scheme. Fig. 5 (middle) shows the refinement of a regular triangular mesh after a step of that scheme.

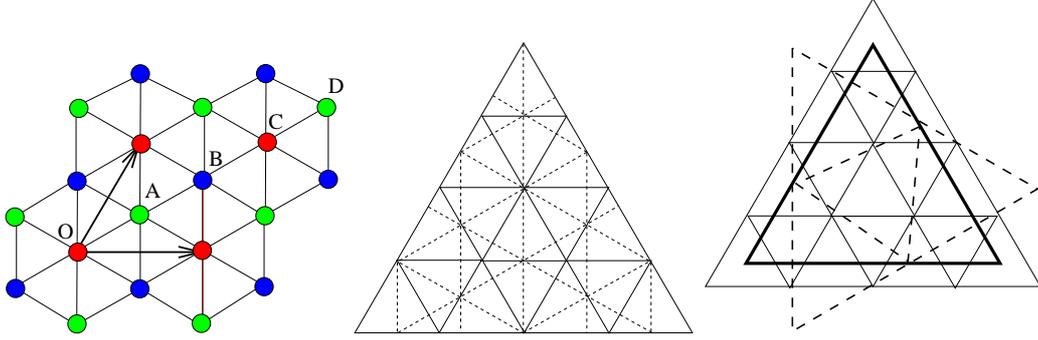


Figure 5: Left: The edge OB gives the scheme with the lowest possible arity. The edge OC gives the $\sqrt{3}$ -scheme and the edge AD gives a dual $\sqrt{3}$ -scheme. Middle: A scheme with arity $\frac{2\sqrt{3}}{3}$. Right: The dual $\sqrt{3}$ -scheme after one step (dotted line) and after two steps (thin line).

The lowest distance, greater than 1, between two red points is $\sqrt{3}$ and gives the $\sqrt{3}$ -scheme [10]. It has the lowest arity among the primal schemes.

The lowest distance, greater than 1, between two green or between two blue points is again $\sqrt{3}$. It gives a dual $\sqrt{3}$ -scheme which has the lowest arity among the dual schemes.

Notice that if we displace the new inserted vertices of the original $\sqrt{3}$ -scheme by $\frac{1}{3}$ in any of the six directions of the lattice, we get the new inserted vertices of the dual $\sqrt{3}$ -scheme. That also means that all the new points of the dual $\sqrt{3}$ -scheme are introduced on the edges of mesh. Fig. 5 (right) shows the mesh after one and two iterations of the dual $\sqrt{3}$ -scheme.

There are several difficulties in extending the dual $\sqrt{3}$ -scheme, or the scheme with arity $\frac{2\sqrt{3}}{3}$, to cover the irregular case. There is an obvious extension when the mesh is bipartite but this extension fails in the general case when there are vertices with odd valency.

Nevertheless, in the regular case there is a compensation for the lower symmetry of the dual $\sqrt{3}$ -scheme. Inserting the new points on the edges of the old mesh as weighted means of the ends of the edges, will give a $\sqrt{3}$ -scheme with polygonal support [9]. This contrasts to the primal $\sqrt{3}$ -scheme which has fractal support.

5 Quad lattices

In Sloan et. al [14] [13] there is a detailed study of the inclusion $L_0 > L_1$, between orthogonal lattices, in arbitrary dimension, for the purposes of numerical approximation of multiple integrals. If we add the centres of L_1 , the new point set is also an orthogonal lattice L , although its two generators have length $\frac{\sqrt{2}}{2}$ and are rotated by $\frac{\pi}{4}$. Nevertheless, the complete classification [13] holds and gives all the possible subdivision schemes under our assumptions.

Here we will restate that classification in a language analogous to that of the triangular meshes. We use the coordinate system defined by the two generators of the lattice L_1 . The situation is simpler because all the centres are the same for our purposes. Again, one edge P_0P_1 of the lattice L_0 defines it and we separate three cases

1. If both P_0, P_1 are vertices of L_1 , then the lattice L_0 is a subset of L_1 , and the corresponding scheme is primal. We may assume, up to a translation of L_1 , that P_0 is the origin O . Then the position of P_1 defines the scheme up to a rotation of order 4 through O . As P_1 is also a vertex of L_1 its coordinates have the form

$$(n, m), \quad n, m \in \mathbf{Z} \quad (13)$$

The equivalence induced by the rotation identifies the points

$$(n, m), (-m, n), (-n, -m), (m, -n). \quad (14)$$

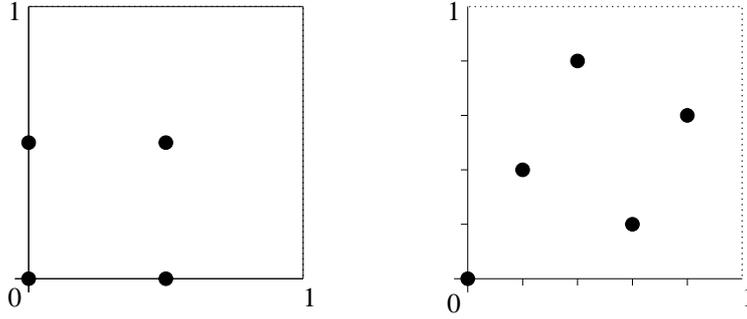


Figure 6: Left: The classic binary refinement. Right: The refinement rule is characteristic because no two points have the same x or y coordinate [14].

2. If both P_0, P_1 are centres of L_1 , then all the points of L_0 are centres of L_1 and the corresponding scheme is dual. We may assume, up to a translation of L_1 , that P_0 is at $(\frac{1}{2}, \frac{1}{2})$ and the position of P_1 defines the scheme up to a rotation of order 4 through P_0 . As P_1 is also a centre of L_1 its coordinates have the form

$$(n + \frac{1}{2}, m + \frac{1}{2}), \quad n, m \in \mathbf{Z} \quad (15)$$

3. Finally if P_0 is vertex and P_1 is centre, we may assume P_0 at the origin, and the position of P_1 is given by

$$(n + \frac{1}{2}, m + \frac{1}{2}), \quad n, m \in \mathbf{Z} \quad (16)$$

The scheme is mixed primal-dual and the coordinates of P_1 define it up to the equivalence (14). Around any of the quadrilaterals of L_0 we get the sequence VFVF of points of L_1 .

Fig. 6 shows two examples of primal schemes, a classic one corresponding to binary subdivision, and one that does not correspond to any known scheme. We used the lattice duality, that is, we normalised the lattice L_0 rather than the L_1 , to have figures similar to that in Sloan et. al [14].

It worth noticing that some of the analysis of the lattice rules for numerical integration may be carried over to subdivision. There are similarities between numerical integration and subdivision, as both processes involve the calculation of an average from a discrete set of points. For example we would expect that the good behavior of the 5-point lattice rule of Fig. 6 (right), caused by the good distribution of the x and y coordinates, will be manifested in the corresponding subdivision scheme. Such a scheme would be a quad version of the (2,1) triangular scheme proposed in Alexa [1].

6 Hexagonal meshes

Here we will use the same methods to classify the regular subdivision schemes on hexagonal meshes. Although the regular hexagonal meshes (honeycombs) are rarely used in practice, their classification is important not only for the sake of completeness but also because of their relation with triangular schemes through duality. Notice that the vertices of a hexagonal mesh do not form a lattice, that is, they do not form an additive subgroup of \mathbf{R}^2 .

So, we consider a regular hexagonal mesh M_0 and we want to find another regular hexagonal mesh M_1 corresponding to the next step of the subdivision process. We will make the same assumption for the acceptable positions for the new vertices. That is, the critical points are the vertices and the centres of M_0 , or equivalently, the points of M_0 fixed by a rotational symmetry of order 3 or 6. These points form a triangular lattice L , see Fig. 3 (right). As M_1 is not a lattice

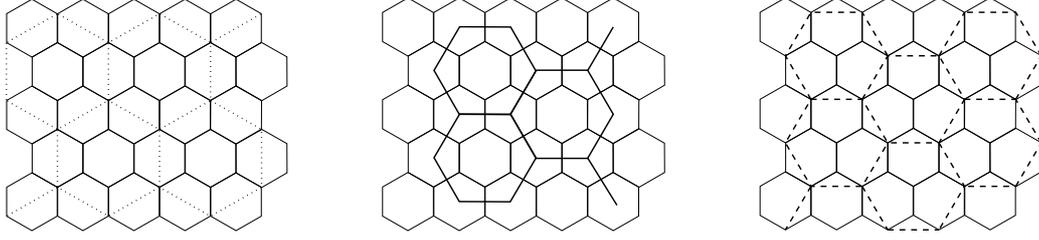


Figure 7: Left: A hexagonal scheme of arity 2 proposed in Dyn et. al [6]. Its dual is the Loop subdivision. Middle: A hexagonal scheme of arity $\sqrt{3}$. It is known from its application in Discrete Mathematical Chemistry [8]. As a subdivision scheme was studied in [3]. Right: A second scheme with arity $\sqrt{3}$. Its dual is a dual $\sqrt{3}$ -scheme with polygonal support.

we will use the generators of L for our coordinate system. We assume that the origin O is at a vertex of M_1 and the point $(1,0)$ at a centreface of M_1 .

It is not difficult to find all the regular hexagonal meshes with vertices on L . Every edge P_0P_1 connecting two vertices of L gives rise to a regular hexagonal mesh. Indeed, all the critical points are fixed by a rotation of order 3 or 6, which is divisible by 3, and so we can rotate P_0P_1 through $\frac{2\pi}{3}$ about any of its ends, and continuing this way create the new hexagonal mesh. Again we separate three cases.

1. Both P_0, P_1 are vertices of M_1 . Using symmetries of M_1 we may assume that P_0 is the origin. Then using (7) we find that the position of P_1 is

$$(n, m), \quad n, m \in \mathbf{Z}, \quad n - m \equiv 0, 2 \pmod{3}. \quad (17)$$

The equivalence induced by the rotation of order 3 around the origin identifies the points

$$(n, m), (-n - m, n), (m, -n - m). \quad (18)$$

2. If the P_0, P_1 are centrefaces then the scheme is dual. We may assume that P_0 is at $(1,0)$. Up to the rotation through $(1,0)$ by $\frac{2\pi}{3}$ the scheme is defined by the position of P_1 , which is

$$(n, m), \quad n, m \in \mathbf{Z}, \quad n - m \equiv 1 \pmod{3} \quad (19)$$

3. If P_0 is a vertex and P_1 a centreface, we use a symmetry of M_1 to send P_0 to the origin and the possible positions of P_1 are given by

$$(n, m), \quad n, m \in \mathbf{Z}, \quad n - m \equiv 1 \pmod{3}. \quad (20)$$

In that case, around any hexagon of M_0 we get a sequence VFVVFV of points of M_1 .

Fig. 7 shows some examples of hexagonal schemes with low arity.

7 Summary

Our classification allows to describe a scheme in the following steps.

1. A selection of the underlying regular mesh type
 - Triangular (T)
 - Quadratic (Q)
 - Hexagonal (H)

2. A selection of type of the scheme

- Primal (P)
- Dual (D)
- Mixed (M)

3. Finally we need the edge P_0P_1 as it was defined in the previous sections. For brevity of notation we will not give the coordinates of P_0 as we know that they are $(0,0)$ in the cases TP, TM, QP, QM, HP, HM, and $(\frac{1}{3}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2})$, $(1, 0)$ in the cases TD, QD, HD, corresponding. So, we need only the coordinates of P_1 . The tables show these coordinates and the rotation that identifies equivalent points.

	P_1 – coordinates	Rotation	
		Centre	Angle
TP	(n, m)	$(0, 0)$	$\frac{\pi}{3}$
TD	$(n + \frac{1}{3}, m + \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3})$	$\frac{2\pi}{3}$
TM	$(n + \frac{c}{3}, m + \frac{c}{3}), c = 1, 2$	$(0, 0)$	$\frac{\pi}{3}$
QP	(n, m)	$(0, 0)$	$\frac{\pi}{2}$
QD	$(n + \frac{1}{2}, m + \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$\frac{\pi}{2}$
QM	$(n + \frac{1}{2}, m + \frac{1}{2})$	$(0, 0)$	$\frac{\pi}{2}$
HP	$(n, m), n - m \equiv 0, 2 \pmod{3}$	$(0, 0)$	$\frac{2\pi}{3}$
HD	$(n, m), n - m \equiv 1 \pmod{3}$	$(1, 0)$	$\frac{2\pi}{3}$
HM	$(n, m), n - m \equiv 1 \pmod{3}$	$(0, 0)$	$\frac{2\pi}{3}$

Notice that in the case H the coordinate system is essentially different from the T, Q cases. Also, it is obvious from the above tables that alternative, more economic, codifications are possible. For example, the added fractions in the coordinates of the point P_1 in the cases TD, QD, QM can be omitted as they convey no information.

With this abbreviated notation the familiar Doo-Sabin [5] scheme follows the refinement pattern QD(2,0), the Catmull-Clark [2] is QP(2,0), the Loop [11] and the Butterfly [7] are TP(2,0), and the $\sqrt{3}$ -scheme is TP(1,1). Some further shortening of the notation may be achieved with the unification of P and M cases in a more compact but less instructive notation.

8 Conclusion

We have presented a unified framework for the classification of regular mesh refinement rules, giving some new insights into their mathematics of subdivision schemes. This classification framework encompasses all known subdivision schemes on regular lattices. Given the generality of our approach we were able to find new subdivision schemes, especially concentrating on low arity. We showed that this very low arity usually comes at the expense of symmetry and uniformity.

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