Axioms for univalence

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We show that, within Martin-Löf Type Theory, the univalence axiom [4] is equivalent to function extensionality [4] and axioms (1) to (5) given in Table 1. When constructing a model satisfying univalence, experience shows that verifying these axioms is often simpler than verifying the full univalence axiom directly. We show that this is the case for cubical sets [1].

	Axiom		Premise(s)		Equality		
(1)	*unit*	:			A	$=$	$\sum_{a:A} 1$
(2)	*flip*	:			$\sum_{a:A}\sum_{b:B} C\ a\ b$	$=$	$\sum_{b:B}\sum_{a:A} C\ a\ b$
(3)	*contract*	:	$\mathit{isContr}\ A$	$\to$	A	$=$	1
(4)	*unit*β	:			$\mathit{coerce}\ \mathit{unit}\ a$	$=$	$(a, *)$
(5)	*flip*β	:			$\mathit{coerce}\ \mathit{flip}\ (a, b, c)$	$=$	(b, a, c)

Table 1: $(A, B : \mathcal{U},\ C : A \to B \to \mathcal{U},\ a : A,\ b : B$ and $c : C\,a\,b$, for some universe $\mathcal{U})$

First recall some standard definitions/results in Homotopy Type Theory (HoTT). A type A is said to be *contractible* if the type $\mathit{isContr}(A) :\equiv \sum_{a_0:A}\prod_{a:A}(a_0 = a)$ is inhabited, where $=$ is propositional equality. It is a standard result that singletons are contractible: for every type A and element $a : A$ the type $\mathit{sing}(a) :\equiv \sum_{x:A}(a = x)$ is contractible. We say that a function $f : A \to B$ is an *equivalence* if for every $b : B$ the fiber $\mathit{fib}_f(b) :\equiv \sum_{a:A}(f\ a = b)$ is contractible. Finally, we can define a function $\mathit{coerce} : (A = B) \to A \to B$ which, given a proof that $A = B$, will coerce values of type A into values of type B.

The axioms in Table 1 all follow from the univalence axiom. The converse is also true. The calculation on the right shows how to construct an equality between types A and B from an equivalence $f : A \to B$. This proof, and many other results described in this paper, have been formalised in the proof assistant Agda [3]. Details can be found at http://www.cl.cam.ac.uk/~rio22/agda/axi-univ.

$$
\begin{aligned}
A &= \sum_{a:A} 1 && \text{by (1)}\\
&= \sum_{a:A}\sum_{b:B} f\ a = b && \text{by (3) on } \mathit{sing}(fa)\\
&= \sum_{b:B}\sum_{a:A} f\ a = b && \text{by (2)}\\
&= \sum_{b:B} 1 && \text{by (3) on } \mathit{fib}_f(b)\\
&= B && \text{by (1)}
\end{aligned}
$$

The univalence axiom is not simply the ability to convert an equivalence into an equality, but also the fact that this operation itself forms one half of an equivalence. It can be shown (e.g. [2]) that this requirement is satisfied whenever $\mathit{coerce}\ (\mathit{ua}(f, e)) = f$ for every $(f, e) : \mathit{Equiv}\ A\ B$, where $\mathit{ua} : \mathit{Equiv}\ A\ B \to A = B$ is the process outlined above. In order to prove this we use axioms *unit*β and *flip*β. Had we derived *unit* and *flip* from univalence, these properties would both hold. Note that we need no assumption about *contract* since, in the presence of function extensionality, all functions between contractible types are propositionally equal.

It is easily shown that *coerce* is compositional, and so we can track the result of *coerce* at each stage to see that coercion along the composite equality $ua(f, e)$ gives us the following:

$$a \quad \mapsto \quad (a,\ *) \quad \mapsto \quad (a,\ f\ a,\ \mathit{refl}) \quad \mapsto \quad (f\ a,\ a,\ \mathit{refl}) \quad \mapsto \quad (f\ a,\ *) \quad \mapsto \quad f\ a$$

Experience shows that the first two axioms are simple to verify in many potential models of univalent type theory. To understand why, it is useful to consider the interpretation of $\mathit{Equiv}\ A\ B$ in a model of intensional type theory. Propositional equality in the type theory is not interpreted as equality in the model's metatheory, but rather as a construction on types e.g. path spaces in models of HoTT. Therefore, writing $[\![X]\!]$ for the interpretation of a type X, an equivalence in the type theory will give rise to morphisms $f : [\![A]\!] \to [\![B]\!]$ and $g : [\![B]\!] \to [\![A]\!]$ which are not exact inverses, but rather are inverses modulo the interpretation of propositional equality, e.g. the existence of a path connecting x and $g(f\,x)$. However, in many models the interpretations of A and $\sum_{a:A} 1$, and of $\sum_{a:A} \sum_{b:B} C\ a\ b$ and $\sum_{b:B} \sum_{a:A} C\ a\ b$ will be isomorphic, i.e. there will be morphisms going back and forth which are inverses up to equality in the model's metatheory. This means that we can satisfy *unit* and *flip* by proving that this stronger notion of isomorphism gives rise to a propositional equality between types.

We also assume function extensionality. Every model of univalence must satisfy function extensionality [4, Section 4.9], but it is often easier to check function extensionality than the full univalence axiom. This leaves the *contract* axiom, which captures the homotopical condition that every contractible space is equivalent to a point. The hope is that the previous axioms should come almost "for free", leaving this as the only non-trivial condition to check.

As an example, consider the cubical sets model presented in [1]. In this setting function extensionality holds trivially [1, Section 3.2]. There is a simple way to construct paths between strictly isomorphic types $\Gamma \vdash A, B$ in the presheaf semantics by defining a new type $P_{A,B}$:

$$P_{A,B}(\rho, i) :\equiv \begin{cases} A(\rho) & \text{if } i = 0 \\ B(\rho) & \text{if } i \neq 0 \end{cases} \qquad \text{(where } \rho \in \Gamma(I), i \in \mathbb{I}(I) \text{ for } I \in \mathcal{C})$$

The action of $P_{A,B}$ on morphisms is inherited from A and B, using the isomorphism where necessary. $P_{A,B}$ has a composition structure [1, Section 8.2] whenever A and B do, whose associated *coerce* function is equal to the isomophism. This construction is related to the use of a case split on $\varphi\rho = 1$ in [1, Definition 15]. Finally, given a type $\Gamma \vdash A$ and using the terminology from [1, Section 4.2], the *contract* axiom can be satisfied by taking $\Gamma, i : \mathbb{I} \vdash \mathit{contract}\ A\ i$ to be the type of partial elements of A of extent $i = 0$. The type $\mathit{contract}\ A\ i$ has a composition structure whenever A does. This construction is much simpler than the *glueing* construction that is currently used to prove univalence, and perhaps makes it clearer why the closure of cofibrant propositions under $\forall$ is required [1, Section 4.1].

References

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