Automated Theorem Proving for Special Functions: *The Next Phase*

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Symbolic Numeric Computation. July 28–31, 2014, Shanghai

1. Resolution Theorem Proving

Automated theorem proving

- combining a **logical calculus** with **syntactic algorithms**
- *Full automation* is convenient, but requires a weak calculus.
  - Booleans + arithmetic (SMT)
  - First-order logic (*resolution*)
- Good for program analysis.

- An *interactive theorem prover*
  - allows the construction of elaborate specifications
  - and formal mathematical proof developments
  - in an **expressive** logic,
  - but reasoning is *laborious*.

Interactive theorem proving

- Typically based on some form of *higher-order logic*
  - Isabelle, HOL4: classical HOL, with polymorphism
  - PVS: a classical but dependently-typed HOL
  - Coq: a constructive type theory
- Used for substantial verification projects
- … And to formalise major results in group theory, logic, mathematical analysis, etc.

The resolution proof procedure

- Objective is to **contradict** the *negation* of the statement to be proved.
  - The negated formula is translated to a *conjunction of disjunctions*.
  - A *clause* is a disjunction of literals: atoms or their negations.
- A **resolution step** combines two clauses to yield a new one.
- Producing the **empty clause** terminates the proof: it is the desired contradiction.

A very simple resolution proof

$$[\exists x\, P(x) \vee \exists x\, Q(x)] \wedge \forall x\, [P(x) \rightarrow Q(f(x))] \rightarrow \exists x\, Q(x)$$

$$P(a) \vee Q(b) \qquad \neg P(x) \vee Q(f(x)) \qquad \neg Q(x)$$

$$Q(b) \vee Q(f(a))$$

$$Q(b)$$

$$\square$$

Applications of resolution

- Highly *syntactic* problems:
  - Robbins conjecture
  - completeness of certain axiom systems
- Also in support of interactive theorem proving (Isabelle's *sledgehammer*)

- Mainstream mathematical problems can't easily be reduced to a few first-order formulas.

However, resolution can be <u>modified</u> to solve a class of problems connected with the real numbers…

2. MetiTarski

Resolution for the real numbers

- *MetiTarski* proves first-order statements involving functions such as exp, ln, sin, cos, $\tan^{-1}$
- … using *axioms* bounding these functions by rational functions
- … and *heuristics* to isolate and remove function occurrences
- integrated with the RCF* decision procedures QEPCAD, Mathematica, Z3

*RCF (real-closed field): a field that's first-order equivalent to the reals

Some easy MetiTarski problems

$$0 < t \wedge 0 < v_f \implies ((1.565 + .313v_f)\cos(1.16t) + (.01340 + .00268v_f)\sin(1.16t))e^{-1.34t} - (6.55 + 1.31v_f)e^{-.318t} + v_f + 10 \geq 0$$

$$0 \leq x \wedge x \leq 1.46 \times 10^{-6} \implies (64.42\sin(1.71 \times 10^6 x) - 21.08\cos(1.71 \times 10^6 x))e^{9.05 \times 10^5 x} + 24.24e^{-1.86 \times 10^6 x} > 0$$

$$0 \leq x \wedge 0 \leq y \implies y\tanh(x) \leq \sinh(yx)$$

Each proved in a few seconds!

the basic idea

Our approach involves replacing functions by *rational function upper or lower bounds*.

We end up with *polynomial inequalities*: in other words, RCF problems

... and first-order formulae involving +, −, × and ≤ (on reals) are **decidable**.

RCF decision procedures and *resolution* are the core technologies.

A simple proof:

$$\forall x \; |e^x - 1| \le e^{|x|} - 1$$

negating the claim

$$e^{|c|} < 1 + |e^c - 1|$$

absolute value (neg)

$$0 \le c \vee e^{-c} < 1 + |e^c - 1|$$

absolute value (neg)

$$1 \le e^c \vee 0 \le c \vee e^{-c} < 2 - e^c$$

lower bound: 1-c ≤ e^{-c}

$$1 \le e^c \vee 0 \le c \vee e^c < 1 + c$$

lower bound: 1+c ≤ e^c

$$1 \le e^c \vee 0 \le c$$

“magic"

$$0 \le c$$

absolute value (pos)

$$e^{|c|} < e^c \vee e^c < 1$$

absolute value (pos), etc.

$$c < 0 \vee e^c < 1$$

lower bound: 1+c ≤ e^c

$$c < 0$$

$$\square$$

What about that Magic Step?

$$1 \le e^c \lor 0 \le c$$

an upper bound for exp(x), for x ≤ 0:

$$e^x \le 2304/(-x^3 + 6x^2 - 24x + 48)^2$$

using that upper bound

$$1 \le 2304/(-c^3 + 6c^2 - 24c + 48)^2 \lor 0 < c \lor 0 \le c$$

eliminating the division

$$(-c^3 + 6c^2 - 24c + 48)^2 \le 2304$$

$$\lor\ (-c^3 + 6c^2 - 24c + 48)^2 \le 0 \lor 0 < c \lor 0 \le c$$

deleting redundant literals

$$0 \le c$$

The key: *algebraic literal deletion*

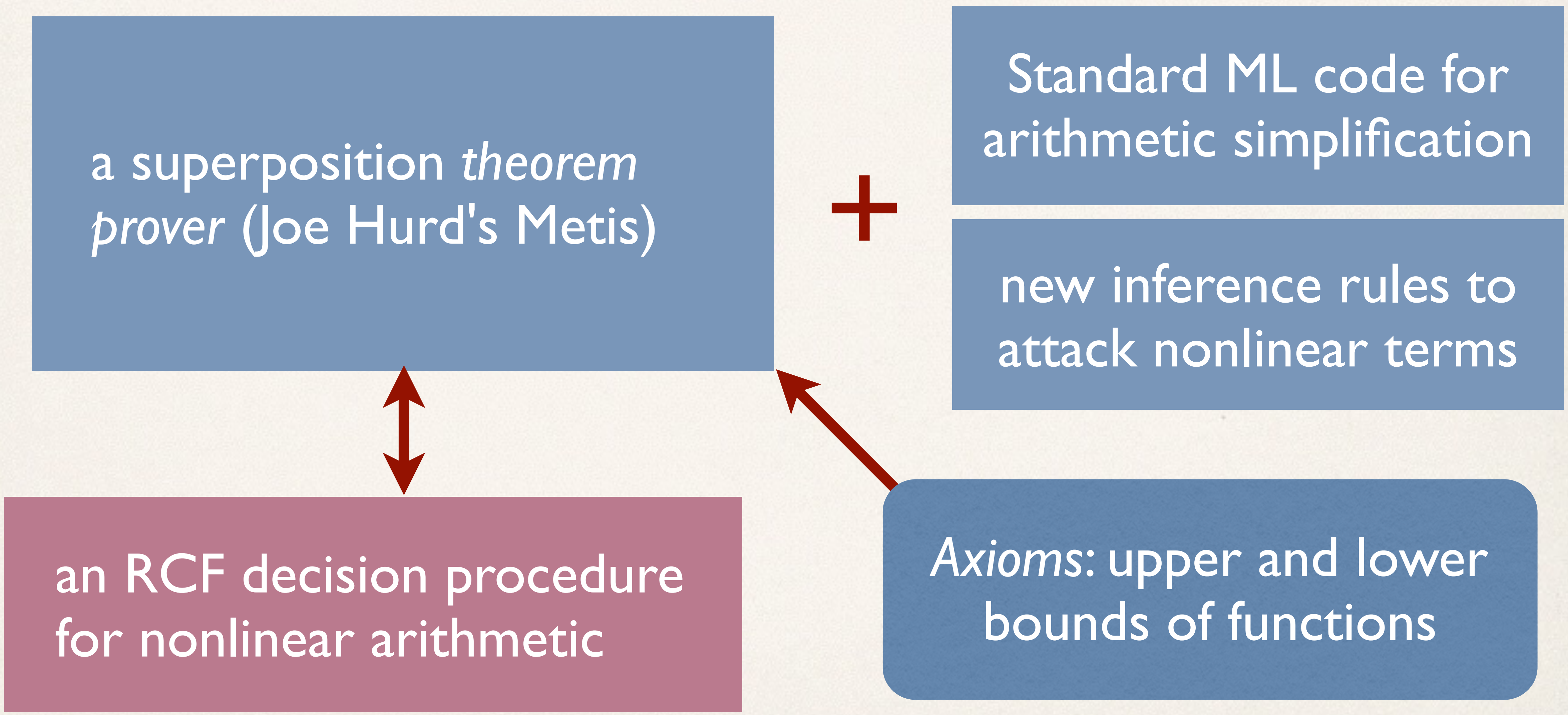
- A list of RCF clauses (algebraic, with no variables) is maintained.
- Every literal of each new clause is examined.
- A literal will be *deleted* if—according to the RCF decision procedure—it is *inconsistent* with its context.
- MetiTarski also uses the decision procedure to detect *redundant* clauses (those whose algebraic part is deducible from known facts).

Examples of literal deletion

- *Unsatisfiable* literals such as $p^2 < 0$ are deleted.
- If $x(y+1) > 1$ has previously been deduced, then $x=0$ will be deleted.
- The context includes the *negations of adjacent literals* in the clause: $z > 5$ is deleted from $z^2 > 3 \lor z > 5$
- … because quantifier elimination reduces $\exists z\, [z^2 \leq 3 \land z > 5]$ to FALSE.
- Or in our example,

$$\exists x\, [x < 0 \land (-x^3 + 6x^2 - 24x + 48)^2) \leq 2304]$$

Architecture



Inherent limitations

- Only **non-sharp** inequalities can be proved.
- Not suitable for developing mathematics:
  - ugly, mechanical proofs
  - … relying on approximations alone, not “insights”
- **Nested** function calls? Difficult.

A few (engineering) applications

- Abstracting non-polynomial *dynamical systems* (Denman)
- KeYmaera linkup: nonlinear *hybrid systems* (Sogokon et al.)
- Collision-avoidance projects for NASA (Muñoz & Denman)

In engineering applications, inequalities typically hold "by accident"

MetiTarski + PVS

- PVS: an interactive theorem prover heavily used by NASA
- … to verify flight control software, etc
- Now PVS uses MetiTarski as an oracle via a **trusted interface**
- … complementing PVS's *branch-and-bound* methods for polynomial estimation
- In NASA's ACCoRD project, MetiTarski has been effective!

3. Upper and Lower Bounds

- MetiTarski works for **any** real-valued function that can be approximated by upper and lower bounds.
- Bounds valid over various intervals, of varying accuracy and complexity, are chosen *automatically*.

Some bounds for ln

- based on the continued fraction for ln(x+1)
- *much* more accurate than the Taylor expansion
- including inaccurate but very simple bounds

$$\frac{x-1}{x} \leq \ln x \leq x - 1$$

$$\frac{(1+5x)(x-1)}{2x(2+x)} \leq \ln x \leq \frac{(x+5)(x-1)}{2(2x+1)}$$

Some bounds for exponentials

$$e^x \geq 1 + x + \cdots + x^n/n! \qquad (n \text{ odd})$$

$$e^x \leq 1 + x + \cdots + x^n/n! \qquad (n \text{ even}, x \leq 0)$$

$$e^x \leq 1/(1 - x + x^2/2! - x^3/3!) \qquad (x < 1.596)$$

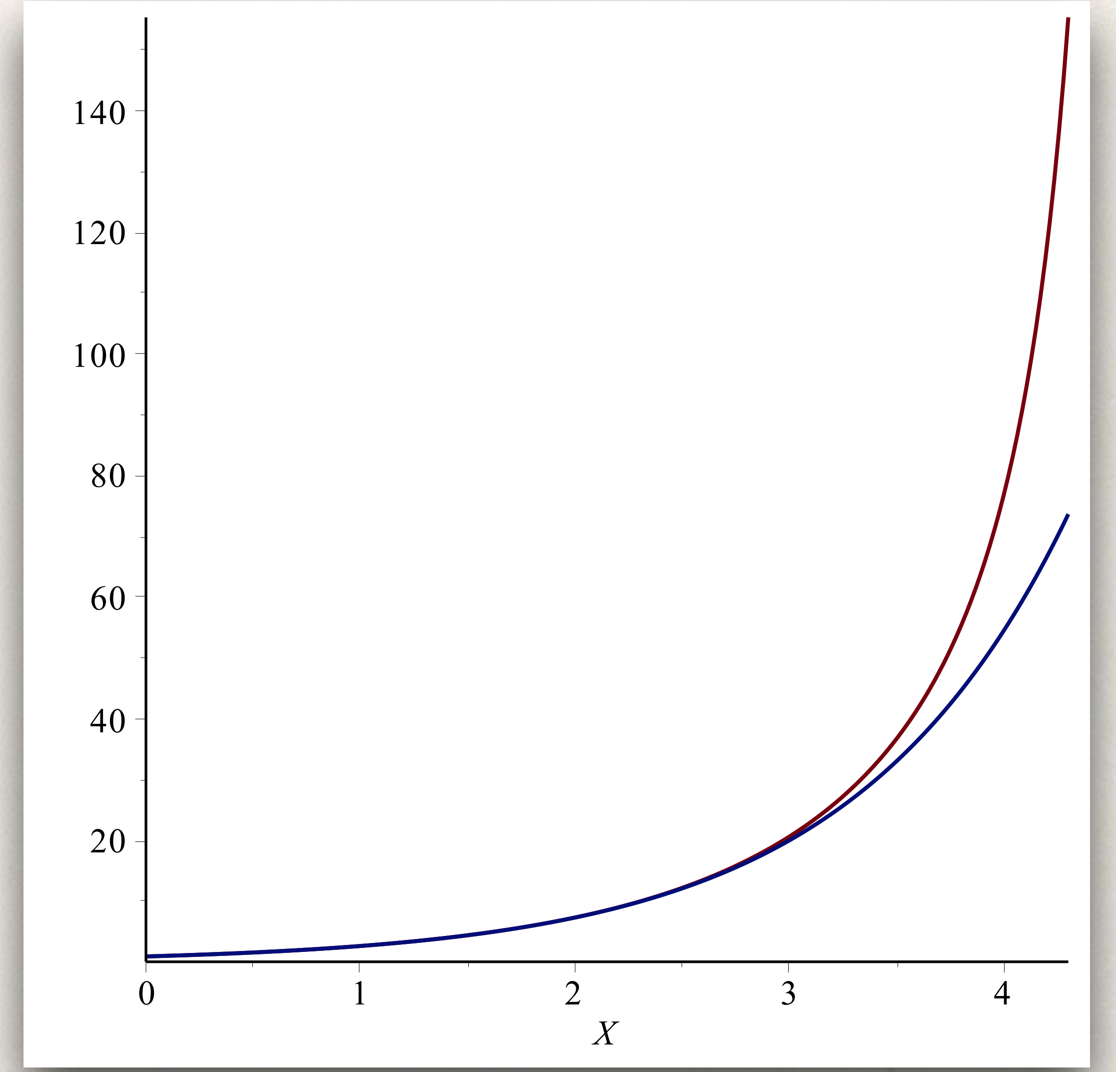
$$e^x \leq -\frac{x^3 + 12x^2 + 60x + 120}{x^3 - 12x^2 + 60x - 120} \qquad (0 \leq x \leq 4.644)$$

From Taylor series,
continued fractions,
identities.

Bounding e^x from above

$$\mathrm{cf3}\,x \triangleq -\frac{x^3 + 12x^2 + 60x + 120}{x^3 - 12x^2 + 60x - 120}$$

- Based on a continued fraction
- **Singularity** around 4.644
- All exponential upper bounds must have singularities!



Verifying MetiTarski's Axioms

- *Taylor series expansions*: already verified (using Isabelle, PVS, etc.) for the elementary functions sin, cos, $\tan^{-1}$, exp, ln.
- *continued fractions*: more accurate; advanced theory
- The axioms for the five transcendental functions have been verified using Isabelle — using simple methods.
- no formalisations of their *general* continued fraction expansions

$\operatorname{cf3} x \geq e^{x} \quad (0 \leq x \leq 4.644)$

By the monotonicity of ln, it's enough to show

$$\ln(\operatorname{cf3} x) \geq x$$

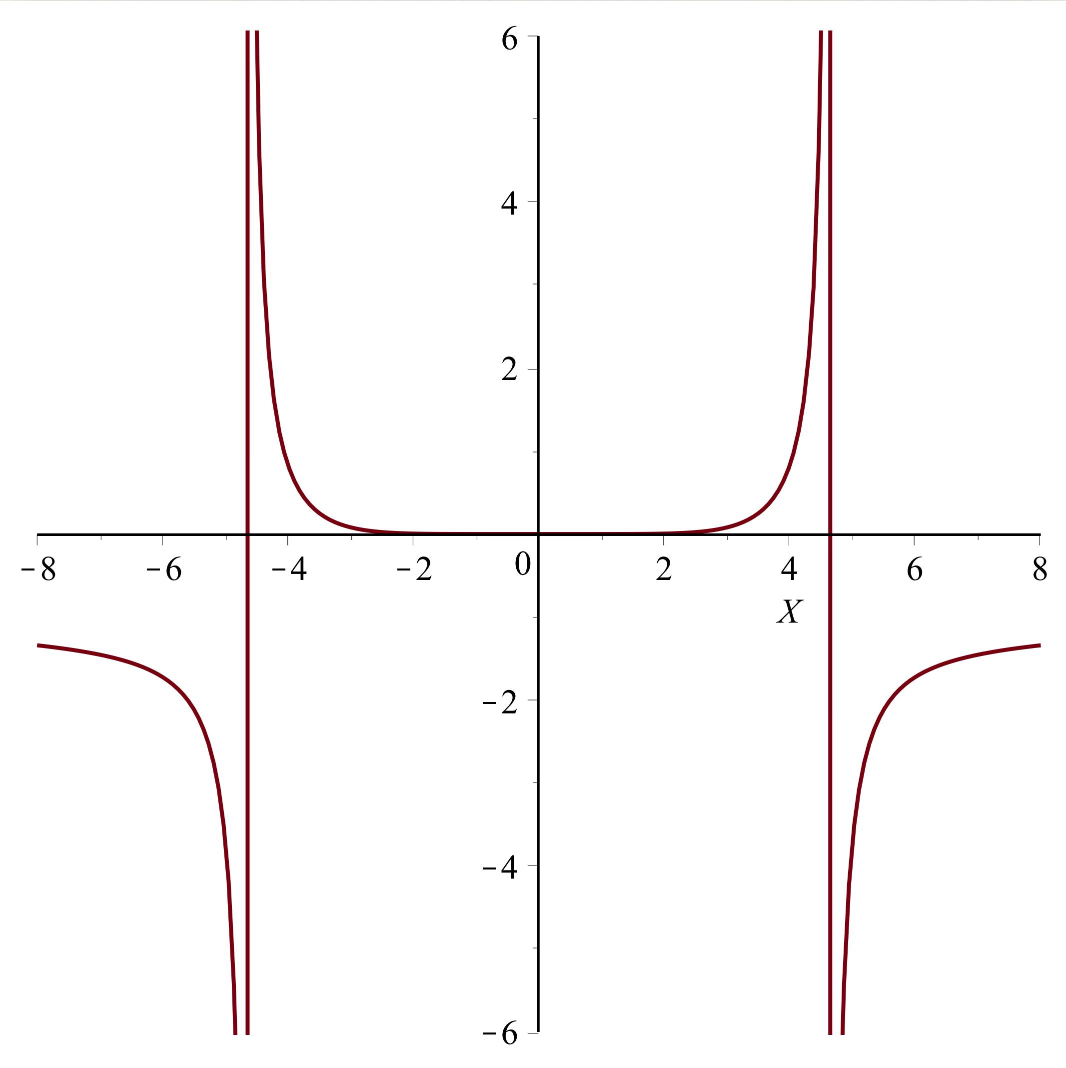
Take the **derivative of the difference:**

$$\frac{d}{dx}[\ln(\operatorname{cf3} x) - x] =$$

$$-\frac{x^6}{(x^3 - 12x^2 + 60x - 120)(x^3 + 12x^2 + 60x + 120)}$$

Plotting that derivative…

- Singularities at ±4.644
- *Nonnegative* within that interval



Continuing the proof sketch

That derivative is positive provided

$$x^3 - 12x^2 + 60x - 120 < 0$$

and in particular if 0 < x < 4.644. And since

$$\mathrm{cf3}(0) = 1 = \exp 0$$

The result follows.

Similar techniques justify a **lower bound axiom:**

$$\mathrm{cf3}\, x \leq e^x \quad (x \leq 0)$$

4. The Next Phase

Correctness concerns

- *floating point arithmetic*:
  - inevitable rounding errors
  - **programmers** responsible for correctness
- *computer algebra systems*: assumptions are made, and **users** are responsible
- *automated theorem provers*:
  - the **system** is responsible for correctness
  - users must be *prevented* from making errors

How can we know that MetiTarski is sound?

MetiTarski soundness questions

- The axioms have been verified.
- MetiTarski produces proofs detailing all first-order reasoning steps.
- Its arithmetic simplification uses straightforward identities.
- So what is left?

Those decision procedure calls.

Cylindrical algebraic decomposition (CAD)

- Given a logical formula involving a set of polynomials in *n* variables
- … **partition** R^n into a finite number of cells
- … such that each polynomial has a *constant sign* on each cell.
- Then quantifiers can be eliminated by picking a member of each cell.

The computational effort is hyper-exponential in *n*!

Simpler: CAD in one variable

Most MT problems are univariate

Hardly any have more than three variables.

In the one-dimensional case, we just need the roots of the polynomials.

CAD within MetiTarski

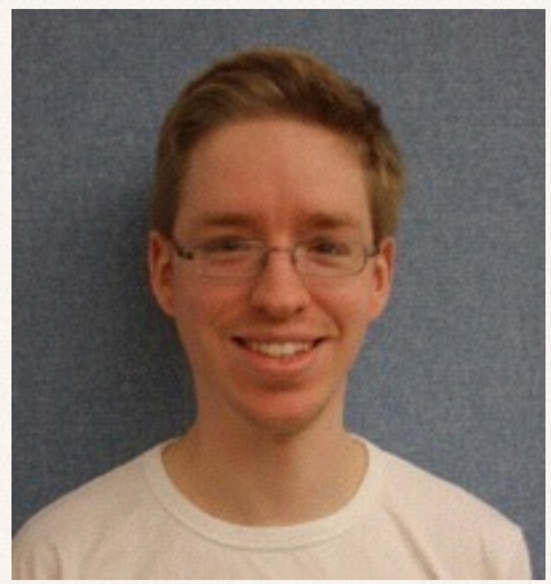
- An experimental extension to MetiTarski solves RCF problems
- … while returning detailed proofs. [Univariate problems only]
- To verify these requires a formalisation of the *Sturm-Tarski theorem*.
- Then MetiTarski could be soundly integrated with interactive theorem provers.

Future aspirations

- MetiTarski works well!
- It will work even better after future improvements to decision procedures.
- ***Interactive theorem proving*** is also effective in mathematical analysis.
- It is time to formalise substantial bodies of complex analysis, real algebraic geometry, etc,
- … and integrate algebraic and analytical reasoning into our theorem-proving tools.

the Cambridge team

James Bridge



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to 2008: Behzad Akbarpour

Acknowledgements

- *Edinburgh Team*: Paul Jackson, Grant Passmore, Andrew Sogokon.
- Assistance from J. H. Davenport, J. Hurd, D. Lester, C. Muñoz, E. Navarro-López, etc.
- Supported by the Engineering and Physical Sciences Research Council [grant numbers EP/C013409/1,EP/I011005/1,EP/I010335/1].

MetiTarski (like Isabelle) is coded in **Standard ML**.