Getting Started With Isabelle

Lecture IV: The Mutilated Chess Board

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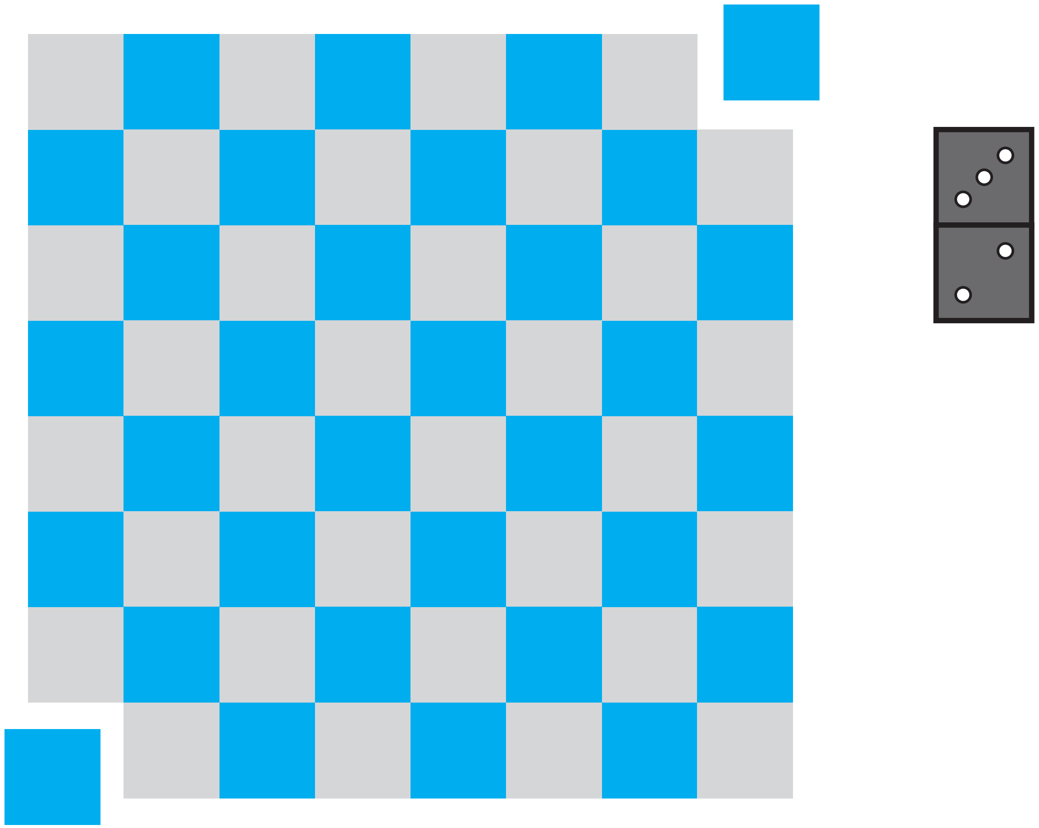
Lecture Outline

- The informal problem
- Inductive definitions
- The Isabelle/HOL specification
- Proof overview

The Mutilated Chess Board

After cutting off the corners, can the board be tiled with dominoes?



The point: find a suitably abstract model.

General Tiling Problems

A tile is a set of points (such as squares).

Given a set of tiles (such as dominoes):

- The empty set can be tiled.
- If t can be tiled, and a is a tile disjoint from t, then the set $a \cup t$ can be tiled.

For A a set of tiles, inductively define $\mathbf{tiling}(A)$:

```
consts     tiling :: "'a set set => 'a set set"
inductive "tiling A"
  intrs
    empty  "{} : tiling A"
    Un     "[| a: A;  t: tiling A;  a <= -t |]
            ==> a Un t : tiling A"
```

Inductive Definitions in Isabelle/HOL

We get (proved from a fixedpoint construction)

- rules `tiling.empty` and `tiling.Un` for making tilings
- rule `tiling.induct` to do induction on tilings:

```
[| xa : tiling A;
   P {};
   !!a t. [| a : A;  t : tiling A;  P t;  a <= - t |]
          ==> P (a Un t) |]
   ==> P xa
```

If property P holds for {} and if P is closed under adding a tile, then P holds for all tilings.

Example: The Union of Disjoint Tilings

If $t, u \in \text{tiling}(A)$ and $t \subseteq \overline{u}$ then $t \cup u \in \text{tiling}(A)$.

base case Here $t = \{\}$, so $t \cup u = u \in \text{tiling}(A)$ by assumption.

induction step Here $t = a \cup t'$, with a disjoint from t'.

Assume that $a \cup t'$ is disjoint from u.

By induction $t' \cup u$ is a tiling, since t' is disjoint from u.

And $a \cup (t' \cup u)$ is a tiling, since a is disjoint from $t' \cup u$.

So $t \cup u = a \cup t' \cup u \in \text{tiling}(A)$.

The Proof Script for Our Example

```
Goal "t: tiling A ==> \
\     u: tiling A --> t <= -u --> t Un u : tiling A";

by (etac tiling.induct 1);
```

perform induction over tiling(A)

```
by (simp_tac (simpset() addsimps [Un_assoc]) 2);
```

change $(a \cup t) \cup u$ to $a \cup (t \cup u)$

```
by Auto_tac;
```

tidy up remaining subgoals

```
qed_spec_mp "tiling_UnI";
```

store the theorem

The Isabelle Theory File

```
Mutil = Main +

consts    tiling ...

consts    domino :: "(nat*nat)set set"
inductive domino
  intrs
    horiz  "{(i, j), (i, Suc j)} : domino"
    vertl  "{(i, j), (Suc i, j)} : domino"

constdefs
  below :: "nat => nat set"
   "below n   == {i. i<n}"

  colored :: "nat => (nat*nat)set"
   "colored b == {(i,j). (i+j) mod 2 = b}"

end
```

dominoes too are inductive!

row/column numbering

Proof Outline

Two disjoint tilings form a tiling.

Simple facts about below: chess board geometry

Then some facts about tiling with dominoes:

Every row of length $2n$ *can be tiled.*

Every $m \times 2n$ *board can be tiled.*

Every tiling has as many black squares as white ones.

If t *can be tiled, then the area obtained by removing two black squares cannot be tiled.*

No $2m \times 2n$ *mutilated chess board (*$m, n > 0$*) can be tiled.*

The Cardinality Proof Script

```
Goal "t: tiling domino ==> \
\     card(colored 0 Int t) = card(colored 1 Int t)";

by (etac tiling.induct 1);
```

perform induction over tiling(A)

```
by (dtac domino_singletons 2);
```

a domino has a white square & a black one

```
by Auto_tac;

by (subgoal_tac "ALL p C. C Int a = p --> p ~: t" 1);
```

lemma about the domino a and tiling t

```
by (Asm_simp_tac 1);
by (blast_tac (claset() addEs [equalityE]) 1);
```

using, and proving, this lemma

Benefits of the Inductive Model

Follows the informal argument

Admits a general proof, not just the 8×8 case

Yields a short proof script:

- 15 theorems
- 2.4 tactic calls per theorem
- 4.5 seconds run time

Other Applications of Inductive Definitions

- Proof theory
- Operational semantics
- Security protocol verification
- Modelling the λ-calculus